1. Let $X$ be the space of polynomials of degree $\leq n$, and let $Y$ be the set of polynomials that are zero at $t_1, \ldots, t_j$, $j \leq n$, $t_i \in \mathbb{R}$. Determine $\dim Y$ and $\dim X/Y$.

2. In Theorem 6 (Lax, p.17) take the interval $I$ to be $[1,1]$, and take $n$ to be 3. Choose three points to be $t_1 = -\alpha$, $t_2 = 0$, and $t_3 = \alpha$.

(i) Determine the weights $m_1, m_2, m_3$ so that

$$\int_I p(t) \text{d}t = m_1 p(t_1) + m_2 p(t_2) + m_3 p(t_3) \quad (4)$$

holds for all polynomials of degree $\leq 3$.

(ii) Show that for $\alpha > \sqrt{3/5}$, all three weights are positive.

(iii) Show that for $\alpha = \sqrt{3/5}$, (4) holds for all polynomials of degree $\leq 6$. 

\[ \text{Sep 20, 2018} \]
3. Let \( P_2 \) be the linear space of all polynomials
\[
p(x) = a_0 + a_1 x + a_2 x^2
\]
with real coefficients and degree \( \leq 2 \). Let \( \xi_1, \xi_2, \xi_3 \) be
three distinct real numbers, and then define
\[
l_i = p(\xi_i), \quad i = 1, 2, 3.
\]
(a) Show that \( l_1, l_2, l_3 \) is a basis for \( P_2 \).
(b) (i) Suppose \( \{e_1, \ldots, e_n\} \) is a basis for a vector
space \( V \). Show that there exist linear functions
\( l_1, \ldots, l_m \) in \( V' \) defined by
\[
l_i(e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]
Show that \( \{l_1, \ldots, l_m\} \) is a basis for \( V' \): it is called
the dual basis.
(c) Find the basis \( \{e_1, e_2, e_3\} \) in \( P_2 \) for which
\( l_1, l_2, l_3 \) above is the dual basis in \( P_2' \).

4. Let \( W_1 \) and \( W_2 \) be subspaces of a vector space \( V \)
such that \( H = W_1 \cup W_2 \) is also a subspace. Show
that one of the spaces \( W_1 \) is contained in the other.

5. (a) Prove that the only subspaces of \( \mathbb{R}^1 \) are \( \mathbb{R}^1 \) and
the zero subspace.
(b) Prove that the only subspaces of \( \mathbb{R}^2 \) are \( \mathbb{R}^2 \), the
zero subspace, or scalar multiples of some fixed vector in \( \mathbb{R}^2 \).
(c) Describe all the subspaces of \( \mathbb{R}^3 \).
6. Let $V$ be the set of real numbers. Regard $V$ as a vector space over the field of rational numbers, with the usual operations. Prove that the vector space is not finite dimensional.