1. Show by Gaussian elimination that the only left null vectors of

\[
M = \begin{pmatrix}
1 & 1 & 2 & 3 \\
1 & 2 & 3 & 1 \\
2 & 1 & 2 & 3 \\
3 & 4 & 6 & 2 \\
\end{pmatrix}
\]

are of the multiples of \( l = (1 \ -2 \ -1 \ 1) \). Then use the fact that for a linear map \( T \), \( R_T^\perp = N_{T^*} \) to conclude that the condition \( 0 = u_4 \ - u_3 \ - 2u_2 \ + u_1 \) is necessary and sufficient to solve the system \( Mx = u \).

**Answer:** To find the left null vectors of \( M \), we perform Gaussian elimination on \( M^T \).

\[
\begin{pmatrix}
1 & 1 & 2 & 3 \\
1 & 2 & 1 & 4 \\
2 & 3 & 2 & 6 \\
3 & 1 & 3 & 2 \\
\end{pmatrix} \to \begin{pmatrix}
1 & 1 & 2 & 3 \\
0 & 1 & -1 & 1 \\
0 & 1 & -2 & 0 \\
-2 & -3 & -7 & \end{pmatrix} \to \begin{pmatrix}
1 & 1 & 2 & 3 \\
0 & 1 & -1 & 1 \\
0 & 0 & -1 & -1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

We may now use back-substitution to find \( l \). First we see \( l_4 = t \) is a free variable. The next row tells us that \( l_3 = -l_4 = -t \). Row 2 says \( l_2 = l_3 - l_4 = 2t \). Row 1 tells us that \( l_1 = -l_2 - 2l_3 - 3l_4 = t \). Altogether we have that \( l = t(1, -2, -1, 1) \), so \( N_{T^*} = \text{span}\{(1, -2, -1, 1)\} \).

We know that \( R_T^\perp = N_{T^*} = \text{span}\{(1, -2, -1, 1)\} \), so

\[\exists x \text{ s.t. } Mx = u \iff u \in R_T \iff l(u) = 0 \forall l \in R_T^\perp \iff t(u_1 - 2u_2 - u_3 + u_4) = 0 \forall t \iff u_1 - 2u_2 - u_3 + u_4 = 0.\]

i.e. \( Mx = u \) is solvable if and only if \( u_1 - 2u_2 - u_3 + u_4 = 0 \).

2. Suppose \( T \in \mathcal{L}(X), \dim X = n \) and let \( B : X \to \mathbb{R}^n \) be an isomorphism such that \( B\alpha_i = e_i, i = 1, \ldots, n \) for some basis \( B = \{\alpha_1, \ldots, \alpha_n\} \) of \( X \). Let \( M = BTB^{-1} \in \mathcal{L}(\mathbb{R}^n) \) and let \( M_{ij} = (Me_j)_i \) be the matrix associated with \( M \) as in Theorem 1 pg 32 (Lax). Show that \( T\alpha_j = \sum_{i=1}^n M_{ij}\alpha_i, i = 1, \ldots, n \). Thus \( M_{ij} \) is the matrix for \( T \) in the basis \( B \).

**Answer:** We use linearity of \( B \) and our definitions to see that

\[T\alpha_j = B^{-1}MB\alpha_j = B^{-1}Me_j = B^{-1}\sum_{i=1}^n (Me_j)_i e_i = \sum_{i=1}^n M_{ij}\alpha_i.\]

This is the definition of \( M \) being the matrix representation of \( T \) in the basis \( B \).

3. Let \( S \) be a linear operator in \( R^2 \) such that \( S^2 = S \) (i.e. \( S \) is a projection). Show that either \( S = 0 \) or \( S = I \) or \( S\alpha_j = \sum_{i=1}^2 A_{ij}\alpha_i j = 1, 2 \) for some basis \( (\alpha_1, \alpha_2) \) for \( \mathbb{R}^2 \), where

\[A = \begin{bmatrix}
1 & 0 \\
0 & 0 \\
\end{bmatrix}.\]

**Note:** I understand this question was hard to read on the sheet, but the question only makes sense if the word after “\( S = I \)” is “or”, and so the homework was marked accordingly.

**Answer:** Since \( S \in \mathcal{L}(\mathbb{R}^2) \), we know that \( \text{rank}(S) = 0, 1, \) or 2. If \( \text{rank}(S) = 0 \), then \( R_S = \{0\} \), and so \( S = 0 \). If \( \text{rank}(S) = 2 \), then \( R_S = \mathbb{R}^2 \). Thus \( \forall y \in \mathbb{R}^2 \) there is an \( x \) such that \( Sx = y \). The fact that \( S^2 = S \) tells us that \( Sy = S^2x = Sx = y \), and so \( Sy = y \) for all \( y \in \mathbb{R}^2 \) and \( S = I \).
If \( \text{rank}(S) = 1 \), then \( \dim \mathcal{N}_S = \dim \mathcal{R}_S = 1 \) and so there exists \( \alpha_1 \neq 0 \) such that \( \mathcal{R}_S = \text{span}\{\alpha_1\} \) and \( \alpha_2 \neq 0 \) such that \( \mathcal{N}_S = \text{span}\{\alpha_2\} \). \( S^2 = S \) tells us that if \( S^2x = 0 \) then \( Sx = 0 \). By question 6 of homework 2, we have that \( \mathcal{R}_S \cap \mathcal{N}_S = \{0\} \). Thus \( \{\alpha_1, \alpha_2\} \) is a linearly independent set of 2 vectors, and so a basis.

To find the matrix of \( S \) in the basis \( \{\alpha_1, \alpha_2\} \), note that \( \alpha_1 \in \mathbb{R}_S \) implies that there is a \( y \) so that \( Sy = \alpha_1 \), then \( S\alpha_1 = S^2y = Sy = \alpha_1 \). Now write \( x = c_1\alpha_1 + c_2\alpha_2 \), and so

\[
Sx = c_1S(\alpha_1) + c_2S(\alpha_2) = c_1\alpha_1.
\]

Thus the matrix for \( S \) is the required \( A \).

4. Let \( X \) be an \( n \)-dimensional vector space over a field \( K \), and let \( \mathcal{B}\{\alpha_1, \ldots, \alpha_n\} \) be a basis for \( X \).

(a) Show that there is a unique linear operator \( T \) on \( X \) such that \( T\alpha_j = \alpha_{j+1} \), \( j = 1, \ldots, n-1 \), and \( T\alpha_n = 0 \). What is the matrix \( A \) of \( T \) in the basis \( \mathcal{B} \). i.e. \( T\alpha_i = \sum_{i=1}^n A_{ij}\alpha_i \), \( i = 1, \ldots, n \)

\[ A_{ij} = \begin{cases} 0 & \text{if } i = j+1 \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]

(b) Prove that \( T^n = 0 \) and \( T^{n-1} \neq 0 \).

\[ A_{ij} = \begin{cases} 0 & \text{if } i = j+1 \\ 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \]

\( A \)

(c) Let \( S \) be any linear operator on \( X \) such that \( S^n = 0 \), but \( S^{n-1} \neq 0 \). Prove that there is a basis \( \mathcal{B}' \) for \( X \) such that the matrix for \( S \) in the basis \( \mathcal{B}' \) is the matrix \( A \) from part a).
Answer: Since $S^{n-1} \neq 0$, there exists an $\alpha_1$ such that $S^{n-1} \alpha_1 \neq 0$. If we let $\alpha_j = S^{j-1} \alpha_1$, I claim that $\mathcal{B}' = \{\alpha_1, \ldots, \alpha_n\}$ is a basis for $X$. Clearly it has the right number of elements, so we need only check linear independence.

Suppose $c_1, \ldots, c_n$ are such that
$$c_1 \alpha_1 + \cdots + c_n S^{n-1} \alpha_1 = 0$$
then applying $S^{n-1}$ to both sides gives
$$c_1 S^{n-1} \alpha_1 + S^n (\alpha_1 + S \alpha_1 + \cdots + S^{n-2} \alpha_1) = 0.$$
By the definition of $\alpha_1$, and the fact that $S^n = 0$, this tells us that $c_1 = 0$. We repeat this process by multiplying by $S^{n-j}$ to show that all of the $c_j$’s are zero and so $\mathcal{B}'$ is a set of $n$ linearly independent vectors, and so a basis for $X$.

This basis also clearly satisfies the property that
$$S \alpha_j = S^j \alpha_j = \alpha_{j+1}, j = 1, \ldots, n-1, \text{ and } S \alpha_n = S^n \alpha_1 = 0 \alpha_1 = 0.$$
Thus $S$ satisfies the same properties that defined $T$ in part a), and so has the same matrix representation.

(d) Prove that $M$ and $N$ are $n \times n$ matrices over $K$ such that $M^n = N^n = 0$ but $M^{n-1} \neq 0$ and $N^{n-1} \neq 0$, then $M$ and $N$ are similar.

Answer: By part c), there exists bases $\mathcal{B}_1$ and $\mathcal{B}_2$ such that writing $M$ and $N$ in those respective bases gives the same matrix representation $A$. Representing these change of basis operations by the matrices $P_1$ and $P_2$, we see that
$$P_1 M P_1^{-1} = A = P_2 N P_2^{-1}$$
$$\Rightarrow M = P_1^{-1} P_2 N P_2^{-1} P_1 = (P_1^{-1} P_2) N (P_1^{-1} P_2)^{-1},$$
and so $M$ and $N$ are similar.

5. Let $W_1$ and $W_2$ be subspaces of a finite-dimensional vector space $X$

(a) Prove that $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$.

Answer: Let $l \in (W_1 + W_2)^\perp$, then for all $w_1 \in W_1$ and $w_2 \in W_2$, $l(w_1 + w_2) = 0$. In particular, choosing $w_2 = 0$ (allowed since $W_2$ is a subspace) gives that $l(w_1) = 0$ for all $w_1 \in W_1$, so $l \in W_1^\perp$. Choosing $w_1 = 0$ similarly shows that $l \in W_2^\perp$. Thus $l \in W_1^\perp \cap W_2^\perp$. Since this is true for all $l \in (W_1 + W_2)^\perp$, we have shown that $(W_1 + W_2)^\perp \subset W_1^\perp \cap W_2^\perp$.

Now suppose $l \in W_1^\perp \cap W_2^\perp$. Then for all $w_1 \in W_1$ and $w_2 \in W_2$, $l(w_1) + l(w_2) = 0$. Using linearity, we see that
$$l(w_1 + w_2) = l(w_1) + l(w_2) = 0,$$
so $l \in (W_1 + W_2)^\perp$. Since this is true for all $l \in W_1^\perp \cap W_2^\perp$, we have that $W_1^\perp \cap W_2^\perp \subset (W_1 + W_2)^\perp$.

Since we have shown containment in both directions, we have the desired equality.
(b) Prove that \((W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp\).

*Answer:* If we let \(Z_1 = W_1^\perp\) and \(Z_2 = W_2^\perp\), then part a) tells us that

\[
(Z_1 + Z_2)^\perp = Z_1^\perp \cap Z_2^\perp
\]

\[
\Rightarrow (W_1^\perp + W_2^\perp)^\perp = (W_1^\perp)^\perp \cap (W_2^\perp)^\perp.
\]

Using the fact that for any subspace \(Y\), \((Y^\perp)^\perp = Y\), we have that

\[
(W_1^\perp + W_2^\perp)^\perp = W_1 \cap W_2
\]

\[
\Rightarrow W_1^\perp + W_2^\perp = (W_1 \cap W_2)^\perp.
\]