Proof of Thm 103.2

Let $A_1$ be a closed sym. exten. of $A$. From Lemma 103.1 we know $D(A_1) = D(A) \oplus A_S$,

where $S$ is an $A$-closed, $A$-symmetric subspace of $\mathfrak{u}_+ \oplus A \mathfrak{u}_-$. If $q \in S$, it can be written uniquely as $q = q_+ + q_-$. Since $S$ is $A$-sym,

$$0 = (A^* q, q) - (q, A^* q)$$

$$= (A^* (q_+ - q_-), q_+ + q_-) - ((q_+ + q_-), A^* (q_+ + q_-))$$

$$= (i(q_+ - i q_-, q_+ + q_-) - (q_+ + q_-), i(q_+ - i q_-))$$

$$= i \left[ - (q_+, q_+) + (q_-, q_-) + (q_+, q_-) - \left( q_+ \right)^* \left( q_- \right)^* \right]$$

$$= 2i \left( q_+, q_- \right) - 2i \left( q_+, q_+ \right)$$

which implies

$$11q_- \| = 11q_+ \|$$

(105.1)

let $B_1 \subset K_1$ denote the projection of $S$ into $\mathfrak{u}_+$,

$$S \ni q = q_+ + q_- \rightarrow q_+$$
Clearly \( \mathcal{B}_1 \) is a linear subspace of \( \mathcal{K}_+ \). Now \( \mathcal{Q}_- \) is uniquely determined by \( \mathcal{Q}_+ \). For if \( \mathcal{Q}_+ + \mathcal{Q}_- \) and \( \mathcal{Q}_+ + \tilde{\mathcal{Q}}_- \) both belong to \( \mathcal{S}_1 \), then
\[
\mathcal{Q}_- - \tilde{\mathcal{Q}}_- = (\mathcal{Q}_+ + \mathcal{Q}_-) - (\mathcal{Q}_+ + \tilde{\mathcal{Q}}_-) \in \mathcal{S}_1.
\]
0 + (\( \mathcal{Q}_- - \tilde{\mathcal{Q}}_- \)) ∈ \( \mathcal{S}_1 \) and hence by (105.1)
\[
\|\mathcal{Q}_- - \tilde{\mathcal{Q}}_-\| = 0 \quad \text{no} \quad \mathcal{Q}_- = \mathcal{Q}_-.
\]
Set
\[
\mathcal{U} \mathcal{Q}_+ = \mathcal{Q}_-, \quad \mathcal{U} \mathcal{Q}_- = 0 \quad \text{if } \mathcal{Q}_+ \in \mathcal{B}_1, \mathcal{U} \mathcal{Q}_- = \mathcal{Q}_-.
\]

Then by (105.1), \( \mathcal{U} \) is a partial isometry from \( \mathcal{K}_+ \) to \( \mathcal{K}_- \) with initial space \( \text{I}(\mathcal{U}) = \mathcal{B}_1 \). We have
\[
\Delta(\Delta_1) = \{ \mathcal{Q} + \mathcal{Q}_+ + \mathcal{U} \mathcal{Q}_+ : \mathcal{Q} \in \mathcal{B}_1, \mathcal{U} \in \text{I}(\mathcal{U}) \}
\]
and
\[
\Delta_1 \cap \mathcal{A}_1^+ \mathcal{A}_1^*\]
\[
\mathcal{A}_1(\mathcal{Q} + \mathcal{Q}_+ + \mathcal{U} \mathcal{Q}_+) = \mathcal{A}_1^* (\mathcal{Q} + \mathcal{Q}_+ + \mathcal{U} \mathcal{Q}_+)
\]
\[
= \mathcal{A} \mathcal{Q} + i \mathcal{Q}_+ - i \mathcal{U} \mathcal{Q}_+
\]

Conversely, let \( \mathcal{U} \) be an isometry from a subspace
Note that $B_i$ is a closed subspace of $K_+$. Indeed suppose that $f_n \in B_i$ and $f_n \rightarrow f \in K_+$. We must show $f \in B_i$ if $f = \overline{q}_+ \text{ for some } q \in S_i$. Now

$$f_n = (\epsilon_n)_+ \text{ for some } \epsilon_n = (\epsilon_n)_+ + (\epsilon_n)_- \in S_i$$

As $\gamma (\epsilon_n)_+ = \epsilon_n$ is Cauchy, $\gamma (\epsilon_n)_- \rightarrow g \in K_-$ must also be Cauchy by (105.1). We have $(\epsilon_n)_- \rightarrow g \in K_-$. Hence

$$\epsilon_n = (\epsilon_n)_+ + (\epsilon_n)_- \rightarrow f + g.$$ 

Also $\epsilon_n \in K_+ \oplus K_-$

$$\therefore \text{ $\mathcal{D}(A^+)$ and } A^+ \epsilon_n = A^+ (\epsilon_n)_+ + A^+ (\epsilon_n)_-$$

$$= \gamma ((\epsilon_n)_+ - (\epsilon_n)_-)$$

$$\rightarrow \gamma (f + g)$$

Hence as $A^+$ is closed, $\overline{q} = f + g \in \mathcal{D}(A^+)$ \lor $A^+ q = A^+ (f + g)$

$$= \gamma (f + g), \quad \text{but } (\epsilon_n - \epsilon_n, q - \epsilon_n)_A = \|q - \epsilon_n\|^2 + \|A^+ q - A^+ \epsilon_n\|^2 \rightarrow 0 \text{ and no, as } S_i \text{ is } A_+ \text{- closed, it follows that } q \in S_i.$$ 

But then $f = \overline{q}_+$, as desired.
of \( K_f \) into \( K_g \) and define \( D(A_1) \) and \( A_1 \) along (104.1) and (104.2). Then it is easy to check that \( D(A_1) \) is an \( A \)-closed, \( A \)-symmetric subspace of \( D(A^*) \), and so by Lemma 103.1, \( A_1 \) is a closed symmetric extension of \( A \).

For \( u \in D(A_1) \), \( u = \psi + \psi_+ + U \psi_+ \)

\[
(i - A_1) u = (i - A_1) \psi + (i - A_1) (\psi_+ + U \psi_+) = (i - A) \psi + 2i U \psi_+..
\]

Now recall that

\[
\ker (i + A^*) = \text{ran } (i - A)^+.
\]

We have \((i + A^*) \psi_+ = (i - i) \psi_+ = 0\) so \( \psi_+ \in \ker (i + A^*) \) and no \( u \psi_+ \in (i - A) \psi \) in particular.

Thus

\[
\ker (i - A_1) = \ker (i - A)^* + \ker (i - A)
\]

and so \( \dim \quad I(U) = \dim \text{ran } \psi < \infty \), we conclude
From (107.1) that \( \text{ran}(c - A) = (\text{ran}(c - A_1)) \oplus (\text{ran} \cup U) \).

\[ n_-(A_1) = n_-(A) - \dim I(U) \]

Similarly \( n_+(A_1) = n_+(A) - \dim I(U) \).

This proves the Theorem. \( \square \)

The following result is immediate.

**Corollary (108.1)**

Let \( A \) be a closed, symmetric operator with deficiency indices \( n_+ \) and \( n_- \). Then,

(a) \( A \) is s. adj \( \iff \) \( n_+ = n_- = 0 \)

(b) \( A \) has s. adj. extensions if and only if \( n_+ = n_- = 0 \). There is a 1-1 correspondence between s. adj. extensions of \( A \) and unitary maps from \( U_+ \) onto \( U_- \).

(c) If either \( n_+ = 0 \) or \( n_- = 0 \) and \( n_2 \neq 0 \)

Then \( A \) has no non-trivial symmetric extensions (such operators are called maximal symmetric).
Proof: (a) follows from Thm 55.1

(b) if $A u$ is a s. adj. extension of $A$, then from

\[ \text{ran} \left( i - A u \right) = \text{ran} \left( i - A \right) \oplus \text{ran} \left( i + A u \right) \]

But as $A u$ is s. adj., $\text{ran} \left( i - A u \right) = \emptyset$ and so

\[ \text{ran} \left( i - A \right) \oplus \text{ran} \left( i + A u \right) = \emptyset \]

In other words,

\[ \text{ran} \left( i + A u \right) = \text{ran} \left( i - A \right) \perp = \ker \left( i + A^* \right) \]

Similarly, we find

\[ \text{I} \left( \right) = \text{ran} \left( i + A u \right) = \text{ran} \left( i - A \right) \perp \]

and so

\[ \text{I} \left( \right) = \text{ran} \left( i - A \right) \perp = \ker \left( i + A^* \right) \]

As $\dim \left( I u \right) = \dim \text{ran} \left( i - A \right) \perp$, we see that

\[ \text{dim} \left( I u \right) = \dim \ker \left( i - A \right) = \dim \ker \left( i + A^* \right) = n_+ \]

Thus $n_+ = n_-$, whether finite or not. Conversely, suppose that $n_+ = n_-$, finite or not. Let $U$ be a unitary map from $K_+$ onto $K_-$, which is possible as $n_+ = n_-$ and let

\[ A u \] be the associated symm. extension of $A$. Then from

\[ \text{ran} \left( i - A u \right) = \text{ran} \left( i - A \right) \oplus \text{ran} \left( i + A u \right) \]

\[ = \text{ran} \left( -A \right) \oplus \text{ran} \left( -A \right) \perp = \emptyset \]

Hence $A u$ is s. adj. This proves (b).
(c) Clearly if \( n_+ = 0 = n_- \) or \( n_+ \neq 0 = n_- \) then there is no (non-trivial) partial isometries \( U \). \( \Box \)

We now illustrate Theorem 10.3.2 with the example

\[ D(T) = \{ f \in L^2(0,1) : f \in AC, f' \text{ c.i.} \} \]
\[ \{ f \in L^2(0,1) : f(0) = f(1) = 0 \} \]

\[ T f = i f', \quad f \in D(T) \]

We showed earlier that \( T \) is a closed, symmetric operator and all its self-adjoint extensions \( \mathcal{S}_\lambda \) have the form

\[ D(\mathcal{S}_\lambda) = \{ f \in L^2(0,1) : f \in AC, f' \text{ c.i.} \} \]
\[ \{ f \in L^2(0,1) : f(0) = e^{i \lambda} f(1) \} \]

\[ e \lambda f = i f', \quad f \in D(\mathcal{S}_\lambda) \]

when \( \lambda \in \mathbb{R} \).
We also showed

\[ D(T^*) = \begin{cases} \mathbb{C} & \text{if } \xi \in \mathbb{R}, \xi' \not\in \mathbb{R}, \\
\mathbb{C} & \text{if } \xi \not\in \mathbb{R}, \xi' \in \mathbb{R} \end{cases} \]

\[ T^* \xi = \xi', \quad \xi \in D(T^*). \]

Now \( \xi \in \mathbb{R}, \xi' \not\in \mathbb{R} \):

\[ (i - T^*) \xi = 0 \]

\[ \Rightarrow \xi - \xi' = 0 \]

\[ \Rightarrow \xi = \xi' \]

\[ \Rightarrow (i - T^*) \xi = 0 \]

\[ \Rightarrow \xi = c e^x \]

\[ \Rightarrow \xi' = i c e^x \]

Similarly, \( \xi \not\in \mathbb{R}, \xi' \in \mathbb{R} \):

\[ \xi = c e^{-x} \]

\[ \xi' = c e^x \]

Thus, \( \eta_+ = \eta_- = 1 \) and \( T \) has s. adj extension.

Let \( \xi_+ = \frac{\sqrt{2}}{\sqrt{e^x - 1}} e^x \) and \( \xi_- = \frac{\sqrt{2}}{\sqrt{e^x - 1}} e^{-x} \)

Note that \( \int_0^1 \xi_+ \xi_+^* dx = 2 \int_0^1 \frac{e^{2x}}{e^x - 1} dx = 1 \)

Similarly, \( \int_0^1 \xi_- \xi_-^* dx = 1 \), so \( \xi_+ \) are normalised

vectors in \( \mathbb{H}_+ \) resp. Now the only isometries of \( \mathbb{H}_+ \) onto \( \mathbb{H}_- \) are maps of the form \( \xi \mapsto \xi' \xi \)

where \( |\xi'| = 1 \). Thus the s. adj. extension \( T \) is assoc.
with $k$ has

$$D(A_{x}) = \psi + \beta \psi_{+} + \beta \psi_{-}, \quad \psi + O(T), \beta \in \mathbb{C}$$

$$A_{x} (\psi + \beta \psi_{+} + \beta \psi_{-}) = i \psi' + i \beta \psi_{+} - i \beta \psi_{-}$$

$$- i \psi' + \beta i \psi_{+} + \beta \psi_{-}$$

$$= i \frac{d}{dx} (\psi + \beta \psi_{+} + \beta \psi_{-})$$

i.e.

$$A_{x} = i \frac{d}{dx} \text{ on } D(A_{x})$$

(112.1)

Finally, note that if $\psi \in D(A_{x})$, then

$$U(10) = \frac{\beta \sqrt{\nu}}{e^{\zeta} - 1} \frac{\nu \psi}{\sqrt{\nu^{2} - 1}} = \frac{(1 + \kappa \epsilon)}{e^{\zeta} - 1} \sqrt{\nu}$$

and

$$U(1) = \frac{\sqrt{2} \beta (e + \kappa)}{\sqrt{e^{\zeta} - 1}}$$

so that

$$U(1) = \frac{e + \kappa}{1 + \kappa \epsilon} U(10) = \lambda U(10)$$

where

$$|\lambda| = \left| \frac{e + \kappa}{1 + \kappa \epsilon} \right| = \left| \frac{e + \kappa}{\kappa \epsilon + e} \right| = \left| \frac{e + \kappa}{\kappa \epsilon + e} \right| = 1$$

Conversely, if $U(1) = \lambda U(10)$, then $U = \psi + \beta \psi_{+}$

$$+ \kappa \beta \psi_{-} \text{ for some } \beta \text{ where } \kappa = \frac{e - e}{1 - \kappa \epsilon}.$$
Thus \( x \mapsto \overline{x} = \frac{x-e}{1-xe} \) maps the "physical" circle to von Neumann's abstract circle.

We note the following important result.

**Definition**

An antilinear map \( C : \mathbb{H} \rightarrow \mathbb{H} \),

\[
C(\alpha u + \beta v) = \overline{\alpha} Cu + \overline{\beta} Cv, \quad u, v \in \mathbb{H}
\]

is called a conjugation if \( \alpha \) is norm preserving

\[
\|Cu\| = \|u\| \quad \forall u \in \mathbb{H}
\]

and

\[
C^2 = 1
\]

**Exercise**

\[ (Cu, Cv) = (u, v) \quad (\text{hint: use polarizatation}) \]

**Theorem (von Neumann's Theorem)**

Let \( A \) be a symmetric operator and suppose that

If a conjugation \( C \) with \( C : D(A) \rightarrow D(A) \) and

\[
A = CA \quad \text{on} \quad D(A)
\]

then \( A \) has equal deficiency indices and therefore has self-adjoint extensions.

**Proof:**

As \( C^2 = 1 \) and \( C D(A) \subseteq D(A) \), we have \( CDCA = D(A) \)

Suppose \( u \in \mathbb{H}_+ \) and \( u \in D(A) \). Then \( \|w\| \leq \|u\| \),
\[ 0 = (\psi_+, (A+i\lambda)\psi_+) = (C\psi_+, C(A+i\lambda)\psi_+) \]

\[ = (C\psi_+, (A-i\lambda)\psi_+) \]

As \( C \) takes \( 0(A) \) onto \( D(A) \). This implies that \( C\psi_+ \in H \).

Thus \( C \) takes \( k_+ \to k_- \). Similarly \( C \) takes \( k_- \to k_+ \).

Since \( C^2 = 1 \) we must have \( Ck_\pm = k_\pm \).

And in particular

\[ \text{dim } k_+ = \text{dim } k_- \]

**Examples:**

\( 1 \)

Let \( H = \mathcal{L}^2(\Omega) \) where \( \Omega \) is a region in \( \mathbb{R}^n \). Let \( D(-A) = \mathcal{D}^\infty(\Omega) \)

\[ -\Delta \varphi = -\sum_{i,j=1}^n \frac{\partial^2 \varphi}{\partial x_i \partial x_j}, \quad \varphi \in D(-A). \]

Let \( C\varphi = \overline{\varphi} \). Then \( -\Delta \) is a symmetric operator on \( D(-A) \) and \( C \) is a conjugation that commutes with \( -\Delta \), \( CA \psi = C(-A) \psi \) on \( D(-A) \).

Thus \( -\Delta \) in \( L^2(\Omega) \) has self-adjoint extensions.
In particular for $n=1$ with $\mathcal{N} = (0,b)$, $0 < b < \infty$

$-\frac{d^2}{dx^2} T \phi_n(x,0,b)$ has p.a.d. extensions. By contrast, we know that $T = \frac{d}{dx}$ is a sym. op. on $L^2_{[0,\infty)}$, but it has no self-adjoint extensions; this is reflected in the fact that the operator $T$ does not commute with complex conjugation.

The fact that $T \Phi_n(x,0,b)$, $b < \infty$, has r.a.d. ext. shows that commuting with a conjugation is not a necessary condition.

Example: Let $AC^2[0,1]$ denote the functions $f \in L^2(0,1)$

such that $f$ is abs. cont., $f'$ is abs. cont. and $f'' \in L^2(0,1)$

Define

$B_0 = \{ f : f \in AC^2(0,1), f(0) = f(1) = 0 \}$

$B_1 = \{ f : f \in AC^2(0,1), f'(0) + f''(0) = 0 \}$

For $a, b$ real,

$B_{a,b} = \{ f : f \in AC^2(0,1), f(0) + f'(0) = 0, b f'(0) + f''(0) = 0 \}$

$D = \{ f \in AC^2[0,1] : f(0) = f(1), f'(0) = f'(1) \}$

$D^* = \{ f \in AC^2[0,1] \}$
and let $T_0, T_{a,b}, T_{a,\infty}, T_{\text{per}}$ and $T$ be the operators $-d^2/dx^2$ with domains $D_0, D_{a,b}, D_{a,\infty}, D_{\text{per}}, D$ respectively.

Then

(a) The operators $T_0, T_{a,b}, T_{a,\infty}, T_{\text{per}}$ and $T$ are closed.

$T_0$ is symmetric but not self-adjoint; its adjoint is $T$.

(b) $T_{a,b}, T_{a,\infty}$ and $T_{\text{per}}$ are self-adjoint extensions of $T_0$.

$-\infty < a < b < \infty$ (there are of course many others!).

Proof: Exercise. Note that once (a) has been established, (b) is easy to verify. Just show in each case that $K_+ = K_- = 303$.

Question: Can we identify each of the extensions $T_{a,b}, T_{a,\infty}$ and $T_{\text{per}}$ with an isometry $U: K_+ \rightarrow K_-$ in Thm 103.2?
We consider, in particular, $T_{0,\omega}$. 

$$k_+ = \{ f \in H^{\beta_0,11} : T_0^x f = i f \}$$

$$k_- = \{ f \in H^{\beta_{-1},11} : T_0^x f = -i f \}$$

Thus, $f \in k_+ \Rightarrow -f'' = i f \Rightarrow f = c e^{\lambda x}$ with $\lambda^2 = i$ i.e. $\lambda = \pm \lambda_+$, $\lambda_+ = e^{-i\pi/4} = \frac{1 + i}{\sqrt{2}}$

Similarly, $f \in k_- \Rightarrow -f''' = -i f$

$$\Rightarrow f = c e^{\lambda x} \quad \text{with} \quad -\lambda^2 = i$$

$$\Rightarrow \lambda = \pm \lambda_-, \quad \lambda_- = e^{i\pi/4} = \frac{1 - i}{\sqrt{2}}$$

Note that $\lambda_+ = \lambda_-$.

Set

\[
\begin{align*}
(117.1) \quad u_+ &= e^{\lambda_+ x} + e^{\lambda_+ (1-x)} \\
&= e^{\lambda_+ x} - e^{\lambda_+ (1-x)} \\
&\quad \text{and} \\
(117.2) \quad u_- &= e^{\lambda_- x} + e^{\lambda_- (1-x)} \\
&= e^{\lambda_- x} - e^{\lambda_- (1-x)}
\end{align*}
\]
Note that

\[(118.1) \quad u_+(x) = u_+(1-x), \quad v_+(x) = -v_+(1-x)\]

and hence

\[(118.2) \quad \{u_+, v_+\} \quad \text{and} \quad \{u_-, v_-\}\]

provide orthogonal bases for \(K_+\) and \(K_-\) respectively.

It is easy to show that if \(f \in D(T_0)\), then as in (112.11), \(T_0 f = -f''\).

We seek a unitary map \(U : K_+ \to K_-\) such that

\[
\Phi U = \{L + \phi_+ + U\phi_+ : \phi_+ \in K_+\} = D_\omega, \quad \text{as we now show, we can construct } U \text{ in the form}
\]

\[(118.3) \quad U u_+ = \chi u_-, \quad U v_+ = \chi v_-\]

\[\text{with } |\chi u_+| = |\chi v_-| = 1. \text{ Such a } U \text{ is unitary by (118.2)}\]

and the fact that \(u_+ = \overline{u}_-\) and \(u_+ = \overline{v}_-\), and hence

\[|u_+| = |u_-| \quad \text{and} \quad |u_+| = |v_-|. \quad \text{The only issue is to show that } U = D_\omega, \quad \text{as s. adjoint operators are}

\[\text{maximally symmetric i.e. if } T \in C \Delta_\omega, \quad \text{with } T = T^*, \quad \text{s. adj and hence symmetric, then } T \in C \Delta_\omega, \quad \text{and } T^* = T, \quad \text{so } T = T_\omega. \quad \text{To show } U \in C \Delta_\omega, \quad \text{it is enough in turn to show that, for suitable } \chi_+, \chi_-, \]

\begin{align*}
\begin{array}{ll}
\text{and} & u^+_1(0) + \chi u^-_1(0) = 0, \quad u^+_1(1) + \chi u^-_1(1) = 0 \\
& v^+_1(0) + \chi v^-_1(0) = 0, \quad v^+_1(1) + \chi v^-_1(1) = 0 \\
\text{but} & u_\pm(1) = u_\pm(0) \quad \text{and} \quad v_\pm(1) = -v_\pm(0) \quad \text{so it is enough to verify}
\end{array}
\end{align*}

\begin{align*}
u^+_1(0) + \chi v^-_1(0) &= (1 - e^{2\lambda}) + \chi (1 - e^{2\lambda}) = 0 \quad \text{and}

v^+_1(0) + \chi v^-_1(0) &= (1 - e^{2\lambda}) + \chi (1 - e^{2\lambda}) = 0 \quad \text{and}
\end{align*}

\begin{align*}
(119,0) \quad & \begin{array}{ll}
\chi u &= -\frac{1 + e^{2\lambda}}{1 + e^{2\lambda}} \quad \chi v &= -\frac{1 - e^{2\lambda}}{1 - e^{2\lambda}}
\end{array}
\end{align*}

As \( \lambda = \overline{x}_+, \) we clearly have \( |\chi u| = |\chi v| = 1 \).

This completes the construction of \( U \) with the desired properties.

Exercise: Find \( U \) for \( t, a, b < \infty \) and \( T \) per.

\[ ||u^+||_c = ||a u^- + b v^-||_c \]
\[ ||v^+||_c = ||c u^- + d v^-||_c \]

\[ u^+_1, v^+_1 = (a u^- + b v^-, c u^- + d v^-) \]

Lecture 10

Another very elegant application of von Neumann's

\begin{align*}
\text{Theorem concerns the Hamburger moment problem with finite moments.}
\end{align*}

Let \( p \) be a pos. measure on \( \mathbb{R} \) and define

\begin{align*}
(119.1) \quad a_n &= \int x^n d\mu(x), \quad n = 0, 1, 2, \ldots
\end{align*}

The \( a_n \) are called the moments of the measure \( \mu \).

The Hamburger moment problem is to determine the conditions

on a sequence of real \( a_n \), \( n = 0 \), so that \( a_n \)

a measure satisfying \((119.1)\)