then $\lambda \in \sigma(A)$. Thus $\bigcup_{n=1}^{N} \operatorname{supp} \mu_n \subset \sigma(A)$ and hence $\overline{\left(\bigcup_{n=1}^{N} \operatorname{supp} \mu_n\right)} \subset \sigma(A)$ as $\sigma(A)$ is closed.

On the other hand, if $\lambda_0 \notin \overline{\bigcup_{n=1}^{N} \operatorname{supp} \mu_n}$, then $\lambda_0 \in B$ = largest open set s.t. $\mu_n(B) = 0$ $\forall n$. It follows in particular that $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \subset B$ for some $\varepsilon > 0$.

It is now easy to show as above (p87 et seq) that for

$$G_{\lambda_0}(\lambda) = \frac{1}{\lambda - \lambda_0} \chi_{|\lambda - \lambda_0| > \varepsilon/2}$$

the bounded operator $G_{\lambda_0}(A)$ provides an inverse for $A - \lambda_0$. Thus $\lambda_0 \in \rho(A)$. This proves Th^m 85.1.

Lecture 8

Exercise 90.1

$$\lambda \in \sigma(A) \iff P_{(\lambda-\varepsilon, \lambda+\varepsilon)} \neq 0 \quad \forall \varepsilon > 0.$$

Exercise 90.2 If $A = A^*$, $\sigma(A) \neq \phi$.

Exercise For $\lambda \in \rho(A)$ $\|(A-\lambda)^{-1}\| = 1/\mathrm{dist}(\lambda, \sigma(A))$.

Definition

Suppose A is s.adj. We say $\lambda \in \sigma_{ess}(A)$, the essential spectrum of A, iff $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is infinite dimensional for all $\varepsilon > 0$ (i.e. $\dim(\mathrm{Ran}\, P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)) = \infty$ $\forall \varepsilon > 0$).

If $\lambda \in \sigma(A)$, but $P_{(\lambda-\varepsilon,\lambda+\varepsilon)}(A)$ is finite dimensional for some $\varepsilon > 0$, we say $\lambda \in \sigma_{disc}(A)$, the discrete spectrum of A.

Exercises

(87.1) $\sigma_{ess}(A)$ is always closed.

(87.2) $\lambda \in \sigma_{disc}(A)$ $\Leftrightarrow$ both of the following hold

(a) λ is an isolated point of $\sigma(A)$ i.e. $\exists \varepsilon > 0$ st $(\lambda-\varepsilon, \lambda+\varepsilon) \cap \sigma(A) = \{\lambda\}$

(b) λ is an eigenvalue of finite multiplicity i.e. $\dim \{\psi \in D(A) : A\psi = \lambda\psi\} < \infty$.

(87.3) ~~Ex~~ (Weyl's criterion). Let $A = A^*$ in $\mathcal{H}$. Then

$\lambda \in \sigma(A) \iff \exists \{\psi_n\} \subset D(A)$ st $\|\psi_n\| = 1$ and

$\|(A-\lambda)\psi_n\| \to 0$ as $n \to \infty$. Furthermore $\lambda \in \sigma_{ess}(A)$ iff

the above $\{\psi_n\}$ can be chosen orthogonal.

(88.1) ~~(d)~~ Stone's formula for the projection valued measures

Let $A = A^*$ in $\mathcal{H}$. Then for any $a < b$.

$$\underset{\varepsilon \downarrow 0}{s\text{-}\lim} \frac{1}{2\pi i} \int_a^b \left[(A-\lambda-i\varepsilon)^{-1} - (A-\lambda+i\varepsilon)^{-1} \right] d\lambda$$

$$= \frac{1}{2}\left[P_{[a,b]} + P_{(a,b)} \right]$$

In particular if $P_{\{a\}} = P_{\{b\}} = 0$ then the

RHS is just $P_{(a,b)}$. Clearly there can only

be a ctble set $\{a_n\}$ st $P_{\{a_n\}} \neq 0$.

Z

Let $A = A^*$ in $\mathcal{H}$.
The following result shows how A generates

a <u>strongly - continuous one - parameter unitary group</u> is a collection of operators $(U(t))_{t \in \mathbb{R}}$ satisfying (a) (b) below.
Let $U(t) = e^{itA} = h_t(A)$, where h_t is the bded

Borel function $e^{it\lambda}$. Then

<u>Th^m 43.1</u>

(a) For each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and

$$U(t+s) = U(t)\,U(s) \quad \forall\ s, t \in \mathbb{R}$$

(b) If $\varphi \in \mathcal{H}$ and $t \to t_0$, then $U(t)\varphi \to U(t_0)\varphi$

(c) For $\phi \in D(A)$, $\dfrac{U(t)\phi - \phi}{t} \to iA\phi$ as $t \to 0$.

(d) If $\lim_{t \to 0} \dfrac{U(t)\phi - \phi}{t}$ $\exists$, then $\phi \in D(A)$

(and the limit $= iA\phi$).

<u>Proof</u>: (a) follows directly from the func. calc. for A.

(b) $$\| (U(t) - U(t_0))\varphi \|^2 = \| (e^{itA} - e^{it_0A})\varphi \|^2$$

$$= \int |e^{it\lambda} - e^{it_0\lambda}|^2 \, d(\phi, P_\lambda \phi) \to 0 \text{ (as } t \to t_0\text{) by dom. conv.}$$

(c) For $\phi \in D(A)$, have

$$\int x^2 \, d(\phi, P_x \phi) = \int x^2 \sum_{n=1}^{N} |\psi(x,n)|^2 \, d\mu_n(x) < \infty$$

Then

$$\left\| \left(\frac{e^{itA} - 1}{t} \right) \phi - iA\phi \right\|^2$$

$$= \int \left| \frac{e^{itx} - 1}{t} - ix \right|^2 d(\phi, P_x \phi)$$

As $\left| \frac{e^{itx} - 1}{t} \right| \le |x|$, we may apply

dom. conv. to conclude that $\frac{e^{itA} - 1}{t} \phi \to iA\phi$.

(d) Have

$$\left\| \frac{e^{itA} - 1}{t} \phi \right\|^2 = \int \left| \frac{e^{itx} - 1}{t} \right|^2 d(\phi, P_x \phi).$$

If $\lim_{t \to 0} \frac{e^{itA} - 1}{t} \phi$ $\exists$, then by Fatou

$0 \le \int x^2 \, d(\phi, P_x \phi)$.

$$= \int \liminf_t \left| \frac{e^{itx} - 1}{t} \right|^2 d(\phi, P_x \phi) \le \liminf_t \int \left| \frac{e^{itx} - 1}{t} \right|^2 d(\phi, P_x \phi)$$

$$= \liminf_t \left\| \frac{e^{itA} - 1}{t} \phi \right\|^2$$

$$= \lim_{t \to 0} \left\| \frac{e^{itA} - 1}{t} \phi \right\|^2 < \infty$$

$\therefore$ $\int x^2 \, d(\phi, P_x \phi) < \infty$ i.e. $\phi \in D(A)$ $\square$.

The following result due to Stone is the converse of Th^m 93.1.

<u>Th^m 95.1</u> (Stone's Theorem)

Let $U(t)$ be a strongly continuous one-parameter unitary group (i.e. $U(t)$ satisfies (a), (b) in Th^m 89.1) on a Hilbert space $\mathcal{H}$. Then, there is a s. adj. operator A on $\mathcal{H}$ s.t. $U(t) = e^{itA}$. $\square$

<u>Proof</u>: See RS VOL I, Th^m VIII.8. $\square$.

Thus we can hope to solve a (formal) eq^n

$$i \frac{du}{dt} + Au = 0 \quad , \quad u(0) = u_0$$

and obtain a reasonable solution, $\|U(t)u_0\| = \|u_0\|$ $\forall t$, iff <u>A is s. adj.</u> !!

<u>Exercise</u> Let $f(t)$ be a measurable function for $t \in \mathbb{R}$ such that $f(t+s) = f(t) + f(s)$ for all $t, s \in B$ where $\mathbb{R} \setminus B$ has measure 0. Show that $f(t) = \gamma t$ for some const. γ.

We now discuss von Neumann's Theory of the s.adj extensions of a symmetric operator A in $\mathcal{H}$ (see RS Vol II, p135 et seq) When does A have s.adj extensions, and how many?

Without loss, we can always assume that A is closed. For if $A \subset S$, $S = S^*$, then S is closed and so $\bar{A} \subset \bar{S} = S$, so S is also an extension of $\bar{A}$.
Also any s.adj extension A_1 of A is clearly a restriction of A^* to some domain $D(A_1)$ with $D(A) \subset D(A_1) \subset D(A^*)$

The following result is closely related to

Th^m 55.1

__Th^m 96.1__ Let A be a closed sym. op. on $\mathcal{H}$. Then

(1a) $\dim \left(\ker(\lambda - A^*)\right)$ is constant in $\mathbb{C}^+$

(1b) $\dim \left(\ker(\lambda - A^*)\right)$ is const in $\mathbb{C}^-$

(2) $\sigma(A)$ is __one__ of the following:

(a) $\overline{\mathbb{C}^+}$

or (b) $\overline{\mathbb{C}^-}$

or (c) $\mathbb{C}$

or (d) a subset of $\mathbb{R}$.

(3) $A = A^* \iff$ (2d) holds.

(4) $A = A^*$ $\Leftrightarrow$ The dimensions in both (1a) and (1b) are zero.

Proof: Let $\lambda = \nu + i\mu$, $\mu \neq 0$. Then, as before, $A \subset A^*$ $\Rightarrow$

(97.1) $$\|(\lambda - A)\varphi\|^2 \geq \mu^2 \|\varphi\|^2$$

$\forall \varphi \in D(A)$. Thus as A is closed, this $\Rightarrow$ $\mathrm{Ran}(\lambda - A)$ is closed in $\mathcal{H}$. Furthermore $\ker(\lambda - A^*) = [\mathrm{Ran}(\bar{\lambda} - A)]^\perp$

Hence $\ker(\lambda - A^*)^\perp = \mathrm{Ran}(\bar{\lambda} - A)$, as $\mathrm{Ran}(\bar{\lambda} - A)$ is (also) closed.

We now show that if $\eta \in \mathbb{C}$ is small enough, $\ker(\lambda - A^*)$ and $\ker(\lambda + \eta - A^*)$ have the same dimension. Let $u \in D(A^*)$ lie in $\ker(\lambda + \eta - A^*)$ with $\|u\| = 1$. Suppose $(u, v) = 0$ $\forall$ $v \in \ker(\lambda - A^*)$.

Therefore, $u \in \mathrm{Ran}(\bar{\lambda} - A)$, by the above, so $\exists$ $\varphi \in D(A)$ st $(\bar{\lambda} - A)\varphi = u$. Thus

$$0 = ((\lambda + \eta - A^*)u, \varphi) = (u, (\bar{\lambda} - A)\varphi) + \bar{\eta}(u, \varphi)$$
$$= \|u\|^2 + \bar{\eta}(u, \phi)$$

This is a contradiction if $|\eta| < |\mu|$, since $\|\varphi\| \leq \frac{1}{\mu}\|u\|$

by (97.4) so

$$\|u\|^2 = -\bar{\eta}(u,\phi) < |\eta|\,\|u\|\,\|\phi\| < \frac{|\eta|}{\mu}\|u\|^2 .$$

Thus for $|\eta| < |\mu|$ there is no $u \in \ker(\lambda+\eta - A^*)$ which is also in $(\ker(\lambda - A^*))^\perp$. Thus any $u \in \ker(\lambda+\eta-A^*)$ has a unique decomposition

$u = u_\parallel \oplus u_\perp$ where $u_\parallel, u_\perp$ lie in $\ker(\lambda - A^*)$ and $\ker(\lambda - A^*)^\perp$ resp.

and $u_\parallel \neq 0$.

Suppose that $u^{(1)}, \dots, u^{(n)}$ are n indep. vectors in $\ker(\lambda+\eta - A^*)$ and let $u_\parallel^{(1)}, \dots, u_\parallel^{(n)}$ be their assoc projection onto $\ker(\lambda - A^*)$. Suppose.

$\sum_{i=1}^n \alpha_i u_\parallel^{(i)} = 0$ for some $(\alpha_1, \dots, \alpha_n) \neq 0$. Now

$\ker(\lambda+\eta-A^*) \ni \sum_{i=1}^n \alpha_i u^{(i)} \neq 0$ as $(\alpha_1, \dots, \alpha_n) \neq 0$. But $\sum_{i=1}^n \alpha_i u_\parallel^{(i)}$ is the projection of $\sum_i^n \alpha_i u^{(i)}$ onto $\ker(\lambda - A^*)$ and hence $\sum_i^n \alpha_i u_\parallel^{(i)} \neq 0$, which is a contradiction. Thus

$u_{11}^{(1)}, \ldots, u_{11}^{(n)}$ are indep. This $\Rightarrow$

(99.1) $$\dim \ker(\lambda + \eta - A^*) \leq \dim \ker(\lambda - A^*)$$

if $|\eta| < |\mu| = |\operatorname{Im} \lambda|$. Conversely, if $|\eta| < \frac{|\mu|}{2}$,

we may apply (99.1) to $(\lambda - A^*)$ to obtain

$$\dim \ker(\lambda - A^*) = \dim \ker((\lambda + \eta) - \eta - A^*)$$

$$\leq \dim \ker(\lambda + \eta - A^*)$$

provided $|\eta| < |\operatorname{Im}(\lambda + \eta)|$. If $|\eta| < |\mu|/2$ then

$$|\operatorname{Im}(\lambda + \eta)| = |(\operatorname{Im} \lambda) + (\operatorname{Im} \eta)| \geq |\mu| - |\operatorname{Im} \eta| \geq |\mu| - |\eta|$$

$> \frac{|\mu|}{2} > |\eta|$. We conclude that

$$\dim \ker((\lambda + \eta) - A^*) = \dim \ker(\lambda - A^*)$$

if $|\eta| < |\mu|/2$. Thus $\dim \ker(\lambda - A^*)$ is

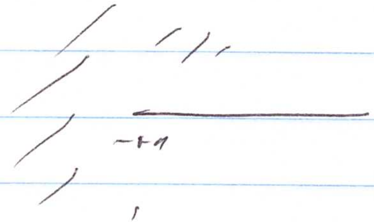
locally constant in $\mathbb{C}^+$ and in $\mathbb{C}^-$ and this

proves (1)

It follows from (97.1) and $\ker(\lambda - A^*) = \operatorname{Ran}(\bar{\lambda} - A)^\perp$

that $\lambda - A$ is a bijection if and only if $\dim \ker(\lambda - A^*) = 0$. By (1a) (1b) we conclude that $\mathbb{C}^+$ and $\mathbb{C}^-$ are either entirely in $\sigma(A)$ or entirely in $\rho(A)$. The fact that $\sigma(A)$ is closed, now $\Rightarrow$ (2). (3) and (4) are now just restatements of Th^m 55.1

<u>Corollary 100.1</u> If A is a closed symm. op. in $\mathcal{H}$ that is semi-bounded i.e. $(A\varphi, \varphi) \geq -M\|\varphi\|^2$ $\forall \varphi \in \mathcal{D}(A)$, for some $0 \leq M < \infty$, then $\dim(\ker(\lambda - A^*))$ is constant for λ in $\mathbb{C} \setminus [-M, \infty)$



<u>Proof</u>: From the proof of Theorem 92.1, we see that the same argument can be carried out concerning the invariance of the dimension for real $\lambda \in (-\infty, -M)$, thus connecting $\mathbb{C}^+$ and $\mathbb{C}^-$. $\square$

Corollary 100.2 If a closed, symmetric operator has at least 1 real # in its resolvent set, then it is s.adj.

Proof: Since $\rho(A)$ is open and contains a pt. in $\mathbb{R}$, it must contain pts in $\mathbb{C}^+$ and in $\mathbb{C}^-$. The corollary now follows from (2) and (3) in Th^m 96.1

Definition Suppose A is a symmetric operator. Let

$$K_+ = \ker(i - A^*) = \operatorname{Ran}(i+A)^\perp$$
$$K_- = \ker(i + A^*) = \operatorname{Ran}(-i+A)^\perp.$$

K_+ and K_- are called the deficiency subspaces of A. The pair of #'s n_+, n_- given by $n_\pm = \dim K_\pm$ resp are called the deficiency indices of A

We now begin the construction of the closed symmetric extensions of A. If B is a closed sym. extension of A, then as noted many times before

(101.1) $$A \subset B \subset B^* \subset A^*$$

We now introduce two new sesquilinear forms on $D(A^*)$:

$$(\varphi, \psi)_A = (\varphi, \psi) + (A^*\varphi, A^*\psi)$$

$$[\varphi, \psi]_A = (A^*\varphi, \psi) - (\varphi, A^*\psi)$$

Sesquilinear $\langle \varphi, \psi \rangle$ means linear in φ, ~~conjugate~~ anti-linear in ψ. A subspace M of $D(A^*)$ st $[\varphi, \psi]_A = 0$ $\forall \varphi, \psi \in M$ will be called A-symmetric. When we refer to a subspace M of $D(A^*)$ as A-closed we mean that if $\varphi \in D(A^*)$, and $\varphi_n \in M \subset D(A^*)$ and $(\varphi - \varphi_n, \varphi - \varphi_n)_A \to 0$, then $\varphi \in M$.

When we refer to a subspaces M_1, M_2 of $D(A^*)$ as A-orthogonal we mean $(\varphi, \psi)_A = 0$ $\forall \varphi \in M_1$ and $\psi \in M_2$.

Lemma 103.1 Let A be a closed, symmetric operator. Then

(a) The closed sym. extensions of A are the restrictions of A^* to A-closed, A-symmetric subspaces of $D(A^*)$ which contain $D(A)$.

(b) $D(A)$, K_+ and K_- are A-closed, mutually A-orthogonal subspaces of $D(A^*)$ and

$$D(A^*) = D(A) \oplus_A K_+ \oplus_A K_-$$

(c) There is a 1-1 correspondence between A-closed, A symmetric subspaces S of $D(A^*)$ which contain $D(A)$ and the A-closed, A-symmetric subspaces S_1 of $K_+ \oplus_A K_-$ given by $S = D(A) \oplus_A S_1$.

Proof: See RS Vol II, pp 138-139.

Our main result is the following.

Theorem 103.2

Let A be a closed symmetric operator. The closed symm. extensions of A are in 1-1 correspondence with the set of partial isometries (in the inner product of H)

of K_+ into K_-. If U is such an isometry with initial space $I(U) \subset K_+$, then the corresponding closed symm. extension A_U has domain

(104.1) $$D(A_U) = \{\varphi + \varphi_+ + U\varphi_+ : \varphi \in D(A),\ \varphi_+ \in I(U)\}$$

and

(104.2) $$A_U(\varphi + \varphi_+ + U\varphi_+) = A\varphi + i\varphi_+ - iU\varphi_+$$

If $\dim I(U) < \infty$, the def. indices of A_U are

$$n_\pm(A_U) = n_\pm(A) - \dim(I(U)).$$

Definition: Recall that $U \in \mathcal{L}(\mathcal{H})$ is an isometry if $\|Ux\| = \|x\|$ $\forall x \in \mathcal{H}$. U is called a partial isometry if U is an isometry on $(\ker U)^\perp$. $I(U) = (\ker U)^\perp$ is called the initial subspace of U and $\operatorname{Ran} U$ is the final subspace.