$\overline{A_0 \restriction D_0} \subset \overline{A_0} = A_0$ and as s.adj. ops are maximally symmetric, we must have

$$A_0 = \overline{A_0 \restriction D_0}$$

$\therefore$ D_0 is a core for A_0

Lecture 6 Finally we show that

(17.20.1) $$D(A_0) = \{ \varphi \in \mathcal{H}_0 : \lambda U\varphi(\lambda) = \lambda \psi(\lambda) \in L^2(d\mu) \}$$

and

(17.20.2) $$(U A_0 U^{-1} \psi)(\lambda) = \lambda \psi(\lambda) \quad \text{for } \psi = U\varphi, \ \varphi \in D(A_0)$$

Let $\varphi = \sum_j \frac{a_j}{A - z_j} e_0 \in D_0 \subset D(A_0)$

Then
$$A_0 \varphi = \sum a_j A \frac{1}{A - z_j} e_0$$
$$= \sum a_j (A - z_j + z_j) \frac{1}{A - z_j} e_0$$
$$= \left(\sum a_j\right) e_0 + \sum a_j z_j \frac{1}{A - z_j} e_0$$

Now $\left(U \frac{1}{1 + i\varepsilon A} e_0 \right)(\lambda) = \frac{1}{1 + i\varepsilon\lambda}$

$\underset{\varepsilon \downarrow 0}{\Rightarrow}$ $U e_0 = 1 \in L^2(d\mu)$

Thus

$$(UA_0\varphi)(\lambda) = \sum a_j + \sum a_i \frac{z_i}{\lambda - z_i}$$

$$= \lambda \sum \frac{a_i}{\lambda - z_i}$$

(17.21.1) ie $(UA_0U^{-1}\psi)(\lambda) = \lambda\psi(\lambda)$, $\psi = U\varphi(\lambda)$.

Suppose $f \in D(A_0)$. Then as D_0 is a core for A_0, $\exists$

$\varphi_n \in D_0$ st $\varphi_n \to f$ and $A_0\varphi_n \to A_0 f$

Thus $\psi_n = U\varphi_n \to Uf$. and $UA_0\varphi_n \to UA_0 f$.

But from (17.21.1), $(UA_0\varphi_n)(\lambda) = \lambda(U\varphi_n)(\lambda)$

Hence, letting $n\to\infty$, we conclude that $\{\lambda U\varphi_n\}$

is Cauchy [in $L^2(d\mu)$] and hence converges to some $g \in L^2(d\mu)$,

and also $UA_0 f = g$. But $\lambda U\varphi_n = \lambda\psi_n$

converges [along a subsequence] almost everywhere to $\lambda Uf(\lambda)$, and so

$g = \lambda Uf$. We conclude in particular that

if $f \in D(A_0)$ then $\lambda Uf(\lambda) \in L^2(d\mu)$ and

(17.21) $UA_0 f(\lambda) = \lambda(Uf)(\lambda)$, $f \in D(A_0)$.

Converseley, suppose that for $f \in \mathcal{H}_0$, $\lambda\, Uf(\lambda) \in L^2(d\mu)$

Then $\exists\, \varphi \in \mathcal{H}_0$ st $U\varphi(\lambda) = (\lambda+i)\, Uf(\lambda)$. Set.

$\tilde{\varphi} = \frac{1}{A_0+i}\varphi$. Clearly $\tilde{\varphi} \in D(A_0)$ and.

$$U\tilde{\varphi}(\lambda) = \frac{1}{\lambda+i}U\varphi(\lambda) = \frac{(\lambda+i)}{\lambda+i}Uf(\lambda)$$

Thus $U\tilde{\varphi}(\lambda) = Uf(\lambda)$ and so $f = \tilde{\varphi} \in D(A_0)$

This proves (17.20.1) (17.20.2).

Denote $d\mu$ by $d\mu_0$.

Remark 17.22: Herglotz functions

The above calculations show that if $F(z)$ is a Herglotz function, i.e. $F(z)$ is analytic and maps $\mathbb{C}_+ \to \mathbb{C}_+$, then $F(z)$ has a Herglotz representation $\boxed{(17.1.1)}$ with $a \geq 0$, $b \in \mathbb{R}$ and $\int d\mu(s)/1+s^2 < \infty$. Moreover for such a & b, $d\mu$ is unique. We stated the Herglotz theorem initially allowing b to be any complex number. This is because one might ask, "Is it possible for a function F given by the RHS of (17.1.1) with general $b \in \mathbb{C}$ to be Herglotz?" The answer is "yes!", and we showed that by modifying $d\mu \to d\tilde{\mu}$ as on p. 77.9, we recover a representation for F with $b \in \mathbb{R}$.

Now suppose $e_1 \perp H_0$ and set

(17.23.1) $$D_1 = \left\{ \sum_{i=1}^{n} \frac{a_i}{A - z_i} e_1 : a_i \in \mathbb{C},\ z_i \in \mathbb{C} \setminus \mathbb{R},\ z_i \neq z_j \text{ for } i \neq j,\ 1 \leq n < \infty \right\}.$$

Suppose $f \in H_0$. Then for $z \in \mathbb{C} \setminus \mathbb{R}$, $\left(f, \frac{1}{A - z} e_1\right)$

$= \left(\frac{1}{A - \bar{z}} f, e_1\right) = 0$ as $\frac{1}{A - \bar{z}} f \in H_0$. Hence $D_1 \perp H_0$ and hence $H_1 = \overline{D_1} \perp H_0$. Moreover

$$A \sum_{i=1}^{n} \frac{a_i}{A - z_i} e_1 = \left(\sum_{i=1}^{n} a_i\right) e_1 + \sum_{i=1}^{n} \frac{a_i z_i}{A - z_i} e_1 \in H_1$$

and so A maps D_1 into H_1. It follows that we can repeat our calculations to show that

$A_1 \upharpoonright D(A) \cap H_1$ is a self-adjoint operator and

There is a probability Borel measure $d\mu_1(\lambda)$ on $\mathbb{R}$ and a unitary map U taking ϕ in H_1 to $\psi(\lambda)$ in $L^2(d\mu_1)$,

(77.24.1) $$\|U\phi\|_{L^2(d\mu_1)} = \|\phi\|_{\mathcal{H}_1}$$

such that

(77.24.2) $$D(A_1) = \{u \in \mathcal{H}_1 : \lambda Uu(\lambda) \in L^2(d\mu_1)\}$$

and

(77.24.3) $$(U A_1 U^{-1}\psi)(\lambda) = \lambda\,\psi(\lambda)$$

for $\psi = U\phi$, $\phi \in D(A_1)$.

We can continue in this way obtaining a sequence of orthogonal spaces $\mathcal{H}_n$, $n \geq 1$, reduce A as above. Two things can happen. Either

(1) $\bigoplus_{n=1}^{N} \mathcal{H}_n = \mathcal{H}$ for some $N < \infty$, and the process stops. Or

(2) $\bigoplus_{n=1}^{k} \mathcal{H}_n \neq \mathcal{H}$ for all k.

Claim In case (2), we necessarily have

$$H = \bigoplus_{n=1}^{\infty} H_n \tag{77.25.1}$$

We use Zorn's Lemma.

Let

$$\mathcal{C} = \left\{ K = \bigoplus_{n=1}^{N} H_n : H_n \perp H_m \text{ for } n \neq m,\ N \leq \infty,\ H_n = \overline{\left\{ \sum_j \frac{a_i}{A - z_i} e_n \right\}} \text{ for some } e_n, \|e_n\| = 1 \right\}$$

Then $\mathcal{C}$ is a partially ordered set with order

$$K = \bigoplus_{n=1}^{N} H_n < K' = \bigoplus_{n=1}^{N'} H'_n$$

if for each n, $H_n = H'_m$ for some m. Suppose $\{K_\alpha\}_{\alpha \in S}$ is a linearly ordered set in $\mathcal{C}$ i.e. for any α, β, either $K_\alpha < K_\beta$ or $K_\beta < K_\alpha$.

Suppose that $\tilde{H}$ is an orthogonal summand for $K_\alpha = \bigoplus_{n=1}^{N^\alpha} H_n^\alpha$, i.e. $\tilde{H} = H_n^\alpha$ for some $n \leq N^\alpha$, and $\hat{H}$ is an orthogonal summand for $K_\beta = \bigoplus_{n=1}^{N^\beta} H_n^\beta$, i.e. $\hat{H} = H_m^\beta$ for some $m \leq N^\beta$. Then if $\alpha < \beta$, then

Clearly either $\tilde{H} = \hat{H}$ or $\tilde{H} \perp \hat{H}$. If $\alpha \neq \beta$, we can suppose without loss that $K_\alpha < K_\beta$. Then $\tilde{H} = H_j^\beta$ for some j, and as $\hat{H} = H_m^\beta$, we again conclude that either $\tilde{H} = \hat{H}$ or $\tilde{H} \perp \hat{H}$.

Let $\mathcal{D} = \{ H^\# : H^\#$ is an orthogonal summand for some K_α, $\alpha \in S \}$

By the above argument, $\mathcal{D}$ is a collection of orthogonal subspaces and hence $\mathcal{D}$ consists of at most a countable set, as $\mathcal{H}$ is separable. Thus $\mathcal{D} = \{ H_j^\# : j \in \mathbb{N} \}$

Set

(77.26.1) $$K = \bigoplus_j H_j^\#$$

Clearly $K \in \mathcal{C}$ and K is an upper bound for $\{K_\alpha\}$, $K_\alpha < K$ for all $\alpha \in S$. Hence by Zorn's Lemma, $\mathcal{C}$ has a maximal element, say $\hat{K}$. Now if $\hat{K} \subsetneqq \mathcal{H}$, then $\exists\, e \in \hat{K}^\perp$, $\|e\|=1$, and we have

$$\mathcal{H}_e \oplus \hat{K} \underset{\neq}{\supset} \hat{K}$$

where $\mathcal{H}_e = \overline{\left\{ \sum_i \frac{a_i}{A-z_i} e \right\}}$, which contradicts the maximality of $\hat{K}$. This establishes the claim (77.25.1).

We are now finally in a position to prove the spectral Th^m 69.1. Given a s.adj. operator A in a separable Hilbert space $\mathcal{H}$, we have proved the existence of Borel (probability) measures $d\mu_n$ on $\mathbb{R}$ $n = 1, \ldots, N \leq \infty$, and a unitary map

$$\mathcal{H} \ni \phi \to U\phi = \psi = \psi_1(\lambda) \oplus \psi_2(\lambda) \oplus \cdots \in \bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$$

$$\|\phi\|_{\mathcal{H}}^2 = \|U\phi\|^2_{\bigoplus_{n=1}^{N} L^2(d\mu_n)} = \sum_{n=1}^{N} \int |\psi_n(\lambda)|^2 d\mu_n(\lambda)$$

Moreover, if $\psi \in \bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$ is such that

$$\bigoplus_{n=1}^{N} \int \lambda^2 |\psi_n(\lambda)|^2 d\mu_n(\lambda) < \infty,$$

then for each n, $\phi_n = U^{-1}\psi_n \in D(A)$ an

$$UA\phi_n(\lambda) = \lambda \psi_n$$

By linearity, for any $j < \infty$,

$$\phi^{(j)} = U^{-1}(\psi_1 \oplus \cdots \oplus \psi_j) \in D(A)$$

and

$$UA\phi^{(j)}(\lambda) = \lambda\, \psi_1(\lambda) \oplus \cdots \oplus \psi_j(\lambda)$$

But as $\psi_j \to \psi$ and $\lambda\psi_j \to \lambda\psi$ in $\bigoplus_{n=1}^{N} L^2(d\mu_n)$

it then follows that $\phi^{(j)} \to \phi = U^{-1}\psi$ and $A\phi^{(j)} \to U^{-1}(\lambda\psi)$

As A is closed it follows that $\phi = U^{-1}\psi \in D(A)$

and $A\phi = U^{-1}\lambda\psi$ i.e. $(UA\phi)_n(\lambda) = \lambda(U\phi)_n(\lambda)$, $n = 1, 2, \ldots$.

On the other hand the multiplication operator Λ on $\bigoplus_{n=1}^{N} L^2(d\mu_n)$

$$(\Lambda\psi)_n(\lambda) = \lambda\psi_n(\lambda)$$

with domain $D(\Lambda) = \{\psi \in \bigoplus_{n=1}^{N} L^2(d\mu_n) : \sum_{n=1}^{N} \int \lambda^2 |\psi_n(\lambda)|^2 d\mu_n(\lambda) < \infty\}$

is s.adj. (exercise), and so $U\Lambda U^{-1}$ with domain $U^{-1}D(\Lambda)$ is self-adjoint in $\mathcal{H}$. But the above calculation show, in particular, that

$$U^{-1}\Lambda U \subset A$$

As self-adjoint operators are maximally symmetric, it follows that $U^{-1}\Lambda U = A$ and so

$$D(A) = \{\phi : \lambda (U\phi)(\lambda) \in \bigoplus_{n=1}^{N} L^2(d\mu_n)\}$$

and

$$(UA\phi)_n(\lambda) = \lambda (U\phi)_n(\lambda)$$

for $\phi \in D(A)$. This completes the proof of the spectral theorem in multiplication operator form.

<u>Remark 77.29.1</u>

Note that $\phi \in D(A)$ if and only if $\lambda(U\phi)(\lambda) = \lambda\psi(\lambda)$ $\in \bigoplus_{n=1}^{N} L^2(d\mu_n)$ ie $\sum_{n=1}^{N} \int \lambda^2 |\psi_n(\lambda)|^2 d\mu_n(\lambda) < \infty$

But then clearly $\phi_n = U^{-1}\psi_n \in D(A)$ and as $\phi = \bigoplus_{n=1}^{N} \phi_n$ we see that if $\phi \in D(A)$, then the projection ϕ_n of ϕ onto $\mathcal{H}_n$ is also in $D(A)$. A priori, this is not obvious. However, if $\phi \in D(A)$, then $(A+i)\phi \in \mathcal{H}$ and we

have
$$(A+i)\phi = \bigoplus_{n=1}^{N} \tilde{\phi}_n \quad , \text{ with } \tilde{\phi}_n \in \mathcal{H}_n .$$

But then
$$\phi = \bigoplus_{n=1}^{N} \frac{1}{A+i} \tilde{\phi}_n$$

and so $\phi = \bigoplus_{n=1}^{N} \phi_n$ with

$$\phi_n = \frac{1}{A+i} \tilde{\phi}_n \in \mathcal{H}_n$$

as desired.

Let $(M, d\mu)$ be a measure space and F a real valued measurable function on M. Let X_F be the operator of multiplication by F on M

$$(X_F f)(m) = F(m) f(m)$$

with domain $D(X_F) = \{ f \in L^2(M, d\mu) : Ff \in L^2(M, d\mu) \}$

Then (exercise) X_F is a self-adjoint operator. The following result tells us that every self-adjoint operator is a multiplication operator on a suitable measure space (in fact a finite measure space).

<u>Corollary 77.31.1</u> (spectral theorem – multiplication operator form)

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there exists a finite measure space $(M, d\mu)$, ~~[illegible]~~ a real valued function $F(m)$, and a unitary map $U : \mathcal{H} \to L^2(M, d\mu)$, so that

$$(UAU^{-1} f)(m) = F(m) f(m)$$

<u>Proof</u>: Choose the cyclic vectors $\{e_n\}$ above such that $\|e_n\| = 2^{-n}$. Let $M = \{\langle \lambda, n\rangle : \lambda \in \mathbb{R},\ 1 \le n \le N\}$, with the natural measure space structure. Define μ on M by requiring that for any n, its restriction to $\{\langle \lambda, n\rangle : \lambda \in \mathbb{R}\}$ is μ_n. Since $\mu(M) = \sum_{n=1}^{N} \mu_n(\mathbb{R}) < \infty$, μ is finite. Also we have for U as above

$$(UAU^{-1} f)(m) = F(m) f(m)$$

, where $F(m) = \lambda$ for $m = \langle \lambda, n\rangle$. $\square$

There is a natural way to define functions of a self-adjoint operater by using Corollary 77.31.1 Given a bounded Borel function h on $\mathbb{R}$ we define (in the notation of 77.31.1)

(77.32.1) $$h(A) = U^{-1} T_{h(F)} U$$

where $T_{h(F)}$ is the operator of multiplication by $h(F(m))$ on $L^2(M, d\mu)$. Using This definition the following Theorem follows easily from Corollary 77.31.1

Lecture 7

__Theorem 77.32.2__ (spectral Theorem - functional calculus form)

Let A be a self-adjoint operator on a separable Hilbert space $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions $\mathcal{B}$ on $\mathbb{R}$ into $\mathcal{L}(\mathcal{H})$ so that

(a) $\hat{\phi}$ is an algebraic $*$- homomorphism

(b) $\hat{\phi}$ is norm continuous, i.e., $\|\hat{\phi}(h)\|_{\mathcal{L}(\mathcal{H})} \leq \|h\|_\infty$