Lecture 5

Example 2

Let $\mathcal{H} = l^2(-\infty, \infty) = \{a_n \in \ell^2 : \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty\}$

Let $L : \mathcal{H} \to \mathcal{H}$ be $(L a)_n = a_{n+1}$

$i.e.$ $L$ shifts to the left. $L^* = \mathbb{R}$ with $(R a)_n = a_{n-1}$

$= \text{shift to right}$. Let $A = R + L = L^* + L = \mathbb{R}^2$

which is a bounded s.a.m. operator

Now map $\mathcal{H}$ into $L^2(0,1)$ by

$U : \{a_n\} \to \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n x}$

Then

$ULU^{-1}$ is mult. by $e^{2\pi i x}$ in $L^2(0,1)$

and

$ULU^{-1}$ is mult. by $e^{2\pi i x}$ in $L^2(0,1)$
Consider $f(x) = \cos x$.

We see that $S_\lambda f = \lambda f$. 

$g(x) = e^{-i\lambda x}$ and $f(1) = e^{ix}f(0)$

$e^{-i\lambda x} = e^{i\lambda x}$ 

$x = \lambda n = -x + 2nx$ 

Now $f_n = e^{-i\lambda nx} = e^{inx}e^{-2\pi inx}$, $n \in \mathbb{Z}$

is clearly a complete, orthonormal set of vectors in $L^2(0, 1)$. Indeed $g \perp f_n$ 

$\langle g, f_n \rangle = \int_0^1 g(x)e^{inx}e^{-inx}dx = 0$ 

And $(f_n, f_m) = \int_0^1 e^{-inx}e^{inx}e^{i(m-x)x}dx = \delta_{n,m}$.
Thus the map \( \phi \mapsto \sum \phi_n (f_n \lambda, k) \) is a unitary map which can then be rewritten (exercise) as giving rise to \( U_\lambda : L^2(0,1) \to L^2(\mathbb{R}, q_{x\lambda}) \) where \( q_{x\lambda}(x) = px \) which turns \( S_\lambda \) into multiplication by \( \lambda \).

Amongst the most important functions \( p(A) \) in the functional calculus for \( A \), are the projection operators mentioned in our first lecture.

\[ P_\lambda = \chi_\lambda (A) \]

where \( \lambda \) is a Borel subset of \( \mathbb{R} \). As \( h \mapsto h(A) \) is \( \ast \)-homomorphism

\[ p_\lambda^2 = \chi_\lambda (A) \chi_\lambda (A) = \chi_{\lambda^2} (A) = \chi_{\lambda^2} (A) = P_\lambda \]

and \( P_\lambda^* = \overline{\chi_\lambda (A)} = \chi_{\lambda} (A) = P_\lambda \)

so \( P_\lambda \) is a self-adjoint projection for each \( \lambda \).

The fact that \( h \mapsto h(A) \) is a \( \ast \)-homomorphism
We now prove the spectral Th~m in multiplication operator form (see Th~m 69.1).

Recall Herglotz's Theorem:

Suppose $F(z)$ is an analytic map from $\mathbb{C}_+$ into $\mathbb{C}_+$. Then $F(z)$ has a representation

\begin{equation}
F(z) = a_3 + b + \int_{\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{s^2 + 1} \right) \, d\mu(s) \tag{77.1.1}
\end{equation}

for some Borel measure $d\mu(s)$ on $\mathbb{R}$ such that

\begin{equation}
\int_{\mathbb{R}} \frac{|d\mu(s)|}{1 + s^2} < \infty \tag{77.1.2}
\end{equation}

and for some constants $a_0$ and $b$.

**Proof:** Exercise. Note that \( \frac{1}{s - z} = \frac{1}{s^2 + 1} \cdot \frac{1}{s - i} \cdot \frac{1}{s + i} \).

Note that if \( z = \mu + i\nu \), \( \nu > 0 \), then

\begin{equation}
\text{Im} \, F(\mu + i\nu) = a_\nu + b \cdot i + \frac{1}{2\pi i} \left[ \int_{\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{s^2 + 1} \right) \, d\mu(s) \right] - \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{s^2 + 1} \right) \, d\mu(s) \tag{77.1.3}
\end{equation}

\[ = a_\nu + b \cdot i + \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{s^2 + 1} \right) \, d\mu(s) = a_\nu + \text{Im} \, b + \frac{1}{2\pi i} \int_{\mathbb{R}} \left( \frac{1}{s - z} - \frac{1}{s^2 + 1} \right) \, d\mu(s) \]
Note that if \( \text{Im} b > 0 \), it clearly is \( \text{Im} F_{\text{utiv}} > 0 \), as it should be. But we note that if

\[
q_1(t) = s
\]

Then

\[
\frac{d}{dt} \sqrt{\frac{ds}{(s^2 - 3)}} = \frac{\sqrt{ds}}{(s^2 - 3)} = \frac{dt}{1 + t^2} = \pi
\]

and so even if \( \text{Im} b < 0 \),

\[
\text{Im} b + \sqrt{\frac{ds}{(s^2 - 3)}} \to 0
\]

If \( q_1 = \delta ds \) with \( s \pi > 1 \text{Im} b \). We will eventually show that \( b \) and \( a \) may be taken to be 0. (see (77.4.3) and (77.11.1))
Now for $A = A^*$ set, fix $e_0 \in \mathbb{B}$ and set

$$F_0(z) = (c_0, \frac{1}{A-z} e_0), \quad z \in \mathbb{C}$$

Then $\frac{1}{(A-z)^{-1}} = (A-z)^{-1}$,

$$\text{Im} F_0(1) = \frac{1}{2i} \left[ (c_0, \frac{1}{A-\bar{z}} e_0) - (c_0, \frac{1}{A-z} e_0) \right]$$

$$= \frac{1}{2i} \left( (c_0, \frac{1}{A-\bar{z}} e_0) - (\frac{1}{A-\bar{z}} c_0, e_0) \right)$$

$$= \frac{1}{2i} \left( (c_0, \frac{1}{A-z} \frac{1}{A-\bar{z}} e_0) - (c_0, \frac{1}{A-\bar{z}} e_0) \right)$$

$$= \frac{1}{2i} \left( (\text{Im} \psi) (c_0, \frac{1}{A-z} \frac{1}{A-\bar{z}} e_0) - (\text{Im} \psi) \frac{1}{A-z} e_0 \right)$$

(7.3.1) $\text{Im} F_0(\overline{z}) = \text{Im} \psi \left( \frac{1}{A-z} \right) e_0 \overline{z}$

Thus by Herglotz Theorem, if $a \geq 0$, $b$ and $c$

Borel meas. $du$, $\int_{\mathbb{R}} \frac{\sqrt{s^2}}{1+5^2} < \infty$, such that

(7.3.2) $\left( c_0, \frac{1}{A-\bar{z}} e_0 \right) = a z + b + \int \left( \frac{1}{5} - \frac{s}{5^2 + 1} \right) \frac{du(s)}{e_0}$
From (72.1.3) and (72.3.1) we have for \( z = u + iv \)

\[
\| A_{-3} z \| = a + \text{Im} b + \int \frac{g(u)}{\sqrt{(s-u)^2 + v^2}}
\]

Now, for any \( f \in \Delta(A) \), \( z = u + iv \), \( v \to 0 \)

\[
\| (A_{-3})^\ast f \| = \| (A_{-u})^\ast f \| = \| g \| + v^2 \| f \|
\]

Thus, for any \( g \in \mathcal{H} \)

\[
\| g \| = \| (A_{-u})^\ast g \| \leq \| A_{-3} \| \| g \|
\]

\[
\| g \| \leq \frac{1}{|\text{Im} z|} \| g \|
\]

\[
\| (A_{-3})^\ast f \| \leq \frac{1}{|\text{Im} z|} \| f \|
\]

Letting \( v \to +\infty \) in (72.4.0) and using (72.4.1) on the LHS, and monotone convergence on the RHS, we conclude that

(77.4.3) \( a = 0 \)
Thus for $y = u + iv$, $v > 0$

\[(77.5.1)\quad \frac{\text{Im} b}{(\alpha - s)^2 + v^2} = \frac{v}{u^2 + v^2} + \int \frac{u}{(s-u)^2 + v^2} \, du \]

Now for $-\infty < \alpha < \beta < \infty$ we have

\[(77.5.2)\quad \int_{\alpha}^{\beta} \frac{1}{u - i v - \alpha} \, du = (\beta - \alpha) \, \text{Im} b + \int_{\alpha}^{\beta} \frac{v}{(u-s)^2 + v^2} \, du\]

\[= (\beta - \alpha) \, \text{Im} b + \int \frac{v}{(u-s)^2 + v^2} \, du\]

Now

\[f(s,v) = \int_{\alpha}^{\beta} \frac{v}{u - i v - \alpha} \, du = \int_{\alpha}^{\beta} \frac{\beta - s}{u - i v} \, du\]

Consider:

(i) $s > \beta - 1$

The $\frac{\alpha - s}{v} < 1 < \frac{\beta - s}{v} < -\frac{1}{v} < -1$ for $0 < v < 1$

and using the elementary inequality

\[\frac{1}{1+t^2} \leq \frac{2}{1+t^2}\]

\[f(s,v) \leq 2 \int \frac{\beta - s}{u - i v} \, dx = 2 \int \frac{\beta - s}{1+t^2} \, dx\]
\[ \frac{1}{1 - \frac{\beta - \delta}{\nu}} \cdot \frac{1}{1 - \frac{\alpha - s}{\nu}} = \frac{1}{1 + \frac{s - \beta}{\nu}} \cdot \frac{1}{1 + \frac{s - \alpha}{\nu}} \cdot \frac{\beta - \alpha}{\nu} \]

\[ \frac{2 \nu^2 (\beta - \alpha)}{(\nu + (\beta - \alpha)) (\nu + s - \alpha)} \leq \frac{2 (\beta - \alpha)}{(s - \beta) (s - \alpha)} \]

which is integrable with \( s \) on \((\beta + 1, \infty)\).

(ii) \( s < \alpha - 1 \).

\[ \frac{1}{\nu} < \frac{\alpha - s}{\nu} < \frac{s - \delta}{\nu} < \frac{\beta - s}{\nu} \]

and using \( \frac{1}{1 + t^c} = \frac{\nu}{(1 + t^c)^c} \), we have

\[ \sigma(s, \nu) \leq \frac{\beta - s}{\nu} \cdot \frac{\nu}{(1 + (\frac{\alpha - s}{\nu})^c)} \]

\[ = \frac{2}{\frac{1}{\nu^2} - \frac{1}{\nu^2}} \cdot \frac{1}{\frac{\nu}{1 + \frac{s - \beta}{\nu}}} \cdot \frac{1}{\frac{\nu}{1 + \frac{s - \alpha}{\nu}}} \cdot \frac{\beta - \alpha}{\nu} \]

\[ = \frac{2 \nu^2 (\beta - \alpha)}{(\nu + (\beta - s)) (\nu + s - \alpha)} \leq \frac{2 (\beta - \alpha)}{(s - \beta) (s - \alpha)} \]

which is integrable with \( s \) on \((-\infty, \alpha - 1)\).

(iii) Finally for \( \alpha - 1 \leq s \leq \beta + 1 \)

\[ |\sigma(s, \nu)| = \left| \frac{\tan \frac{s - \beta}{\nu}}{\nu} - \frac{\tan \frac{s - \alpha}{\nu}}{\nu} \right| \leq \frac{17}{17} \]
It follows that for all \( s \in \mathbb{R} \) and for all \( \omega, \alpha < 1 \)

\[
|f(\omega, s)| \leq g(s)
\]

for some \( g \in L^1(\mathbb{R}) \).

As \( \int_{\omega_0}^\infty (\text{const.} + s - \alpha \cdot \text{term} \cdot \frac{1}{s}) \) \( = 0 \) if \( s > \beta \) or \( s < \alpha \)

\[
= \pi \quad \text{if} \quad \alpha < s < \beta
\]

it follows by the dominated convergence theorem that

\( \gamma \leq \alpha \) and \( \beta \) are not mass points of \( d\mu \)

(There can only be countable \# of such points).

As \( \omega \to \infty \) the LHS of (17.5.5) is given by

\[
(b - \alpha) \text{Im} b + \pi \mu(\alpha, \beta)
\]

But \( b - \alpha = \mu_0(\alpha, \beta) \) where \( \mu_0 \) is Lebesgue measure as.

As the LHS of (17.5.2) depends only on \( A \)

(and of course \( \rho_0 \)) we see that
\( \tilde{\mu} = \mu + \frac{1}{17} (\text{Im} \mu) \mu_0 > 0. \)

is a uniquely determined measure (we use here the fact that \( x \) and \( y \) are dense and \( \forall \mu \), \( \mu([a, b]) = \lim_{\epsilon \to 0} \int \frac{\tilde{\mu}(s)}{|s+\epsilon|^2} ds < \infty \).

We have for \( b \in \mathbb{C}^+ \):

\[
\int \frac{1}{(s^2 + 1)^2} \tilde{\mu}(s) \, ds = \int \left( \frac{1}{s^2 - b} - \frac{s}{s^2 + 1} \right) \tilde{\mu}(s) \, ds + \frac{1}{\pi} \text{Im} \mu \int \frac{s}{s^2 + 1} \, ds
\]

But for \( b \in \mathbb{C}^+ \):

\[
\int \frac{ds}{s^2 - b} = \frac{i}{2} \left( s - \frac{1}{s} \right) \, ds.
\]

\[
= \pi i - \frac{i}{2} \pi i = \pi i
\]

and so

\[
(\alpha_0, \frac{1}{e_0}) = b + \int (\frac{1}{s^2 - b} - \frac{s}{s^2 + 1}) \tilde{\mu}(s) \, ds
\]

\[
= b + \int (\frac{1}{s^2 - b} - \frac{s}{s^2 + 1}) \, d\tilde{\mu}(s),
\]

\[
= -i \text{Im} \mu.
\]
and we conclude that 
\[ F(3) = (c_0, \frac{1}{4 - b}) \]

has a Haarholty representation of the form

\[ (7.10.1) \quad (c_0, \frac{1}{4 - b}) = b + \int \frac{1}{s - b} \, \mu(s) \, ds \]

where \( b \in \mathbb{R} \) and \( \mu \) is a measure with

\[ \int \frac{\mu(s)}{1 + s} < \infty. \]

Returning to (7.5.1) we have for \( z = u + iv, v \neq 0 \)

\[ v^2 \left( A - b \right) \frac{1}{4 - b} c_0 \mu \left( \frac{1}{s - u} + v \right) \]

But the RHS is bounded by (7.4.2) and

hence by monotone convergence, we conclude that

\[ (7.10.2) \quad \int \frac{\mu(s)}{1 + s} < \infty. \]

It follows that we can separate the integrals in (7.10.1) to obtain

\[ (7.10.3) \quad (c_0, \frac{1}{4 - b}) = b + \int \frac{1}{s - 3} \, \mu(s) \]
when \( \hat{b} = b - \int \frac{s}{s^2 + 1} \, dq \).

But we can now let \( s \to \pm \infty \), \( z = u + iv \) to conclude that \( b = 0 \). Thus finally we have the Neumann representation

\[
(77.11.1) \quad (e_0, \frac{1}{A-z} e_0) = \int \frac{q u(s)}{s-\frac{c_0}{c_1}} \, ds, \quad z \in C_+
\]

where \( \int q u(s) < \infty \).

Observe that if \( z \in C_- \), then \( \overline{z} \in C_+ \)

and so

\[
(e_0, \frac{s}{A-z} e_0) = \frac{1}{4-\frac{c_0}{c_1}} (e_0, \frac{s}{A-\frac{c_0}{c_1}} e_0) = \int \frac{q u(s)}{s-\frac{c_0}{c_1}} \, ds = \int \frac{q u(s)}{s-\frac{c_0}{c_1}} \, ds
\]

so \( (77.11.1) \) holds for all \( z \in C \setminus \overline{C} \).
Now let \( D_0 \) be the linear space

\[
D_0 = \sum_{i=1}^{n} \sum_{a_i \in A - \mathbb{R}} \frac{a_i}{3^i} e_0 : a_i e_i \in C \quad \text{for } i \neq 0 \quad 1 \leq n < \infty \quad 3
\]

and let \( A_0 = \overline{D_0} \) be the closure of \( D_0 \).

We will prove later that:

Claim! \( A_0 \), the restriction of \( A \) to \( D(A) \cap A_0 \),

is a self-adjoint operator in \( \overline{D_0} \) and \( D_0 \) is a core for \( A \).

Note that \( \text{core } A_0 \) is called the cyclic subspace generated by \( e_0 \) and \( A \). Indeed, for any \( \varepsilon > 0 \)

\[
q_2 = \frac{1}{1 + \varepsilon A} e_0 \in D_0
\]

Let \( \eta > 0 \) be given; we show that for \( \varepsilon > 0 \) small,

\[
\|q_2 - e_0\| < \eta, \quad \text{Now as the \text{ closure of } } A
\]

is dense, \( \exists f \in D(A) \) such that \( \|f - e_0\| < \eta / 3 \).
\[ \psi_{\epsilon} - \psi_0 = \frac{1}{1 + i \epsilon A} \left( \psi_0 - \psi \right) + \frac{1}{1 + i \epsilon A} \psi_0 - \psi_0 \]

\[ = \psi - \frac{1}{1 + i \epsilon A} \psi_0 + \frac{1}{1 + i \epsilon A} \left( \psi_0 - \psi \right) \]

Thus, using the fact that \[ \frac{1}{1 + i \epsilon A} \leq 1 \] which follows from (77.9.2)

\[ ||\psi_{\epsilon} - \psi_0|| \leq ||\psi - \psi_0|| + \epsilon ||A\psi|| + ||\psi_0 - \psi|| \]

\[ \leq \frac{2}{3} \psi + \epsilon ||A\psi|| \]

Choosing \( \epsilon > 0 \) such that \( \epsilon ||A\psi|| < \frac{\psi}{3} \), we are done.

What we have in fact shown is that

\[ \epsilon \rightarrow \frac{1}{1 + i \epsilon A} \psi_0 \]

Continuing for all \( \psi \).

As \[ ||A\psi|| = \int_{A} \frac{d\mu}{1 + i \epsilon A} \]

by (77.5.1) for \( b = a \), \( \psi = -i \psi \), we conclude that \[ \int_{A} \frac{d\mu}{1 + i \epsilon A} \leq 1 \]

Let \[ \psi = \sum_{i} \frac{a_i}{\lambda_i} \psi_0 \in D_0 \]

Then

\[ ||\psi||^2 = \sum_{i} \frac{a_i}{\lambda_i} \left( \frac{1}{\lambda_i} \sum_{i} \frac{1}{\lambda_i} \right) \]

\[ = \sum_{i} \frac{a_i}{\lambda_i} \frac{1}{\lambda_i} \]

\[ = \sum_{i} \frac{a_i}{\lambda_i} \]
\[ \text{Now } \psi = 3i + 2b; \]

\[ \left( c_0, \frac{1}{\lambda - 3i} - \frac{1}{\lambda - 3j} \right) \frac{1}{\lambda - 3i}; \]

\[ = \int dm(s) \left( \frac{1}{s - 3i} - \frac{1}{s - 3j} \right) \frac{1}{s - 3i}; \]

\[ (77.14.1) \]

Also \( \psi = 3i; \) \( \lambda \psi \)

\[ \left( c_0, \frac{1}{\lambda - 3i} - \frac{1}{\lambda - 3j} \right) \frac{1}{\lambda - 3i}; \]

\[ = \left( c_0, \frac{1}{(\lambda - 3i)^2} c_0 \right) = \delta_{s=3i} \left( c_0, \frac{1}{\lambda - 3i} \right); \]

\[ = 0; \]

\[ = \delta_{s=3i} \int dm(s) \frac{1}{s - 3i}; \]

\[ = \int dm(s) \frac{1}{(s - 3i, 1^t)} \]

\[ = \int dm(s) \frac{1}{(s - 3i, \text{Ks} - 3j)}. \]

In both situations, \((77.14.1)\) holds.

Have,

\[ v = \sum_{i,j} a_i a_j \int dm \frac{1}{s - 3i, s - 3j}; \]
\[ = \int \rho(u) \, \varphi(q) \, \bar{l} \]

where \( \rho(q) = \sum_{i=1}^{n} \frac{a_i}{(\overline{z}, q - 3) i} \).

(17.15.1) \( \| \sum_{i=1}^{n} \frac{a_i}{A - 3 i} \| e = \int \rho(u) \, \varphi(q) \, l_c \).

Suppose \( \varphi \) has 2 representations:

\[ \varphi = \sum_{i \in A - 3 i} a_i \rho = \sum_{i \in A - 3 i} \tilde{a}_i \rho. \]

Then we can assume \( n = \tilde{n} \) and \( 3_i = \tilde{3}_i \).

Then \( 0 = \| \sum_{i=1}^{n} \frac{a_i - \tilde{a}_i}{A - 3 i} \| e = \int \left\| \sum_{i=1}^{n} \frac{a_i - \tilde{a}_i}{(s - 3)i} \right\| e \rho \).

\[ \| \sum_{i=1}^{n} \frac{a_i}{s - 3 i} \| e = \| \sum_{i=1}^{n} \frac{\tilde{a}_i}{s - 3 i} \| e \rho. \]

Thus, the map

\[ \rho_0 \circ \varphi : \sum_{i=1}^{n} \frac{a_i}{A - 3 i} \rho = \sum_{i=1}^{n} \frac{\tilde{a}_i}{s - 3 i} \rho \]

is well-defined and is an isometry by (17.15.1) and hence extends to an isometry \( U \) from \( l_0 \).
into $L^2(d\mu)$.

Let $f \in L^2(d\mu)$. Then a standard argument shows that $f$ converges to a function $F$ with compact support. 

But by Stone-Weierstrass, $U(D_0)$ is $L^\infty$ dense in the class of continuous functions that decay at $\infty$.

Hence, $F \in U(D_0)$ so $\|F_n - F\|_\infty \leq \frac{1}{n}$. But then $\lim_{n \to \infty} \int_{D_0} F_n = 1$.

Then

\[
\|F_n - F\|_2^2 = \int |F_n - F|^2 \, d\mu(s) \leq \frac{1}{n} \int \, d\mu(s) = \frac{1}{n} + \text{error}
\]

Hence $\|F_n - F\|_2 \to 0$ as $n \to \infty$.

Let $F_n = \sum \frac{a^k}{s - \omega} \psi_{\omega}$ where

$F_n = \sum \frac{a^k}{s - \omega}$

Then $\|F_n\| = \|F_n\|_2^2$.

And so as $(F_n) is Cauchy in $L^2(\psi)$. Let $F$ be Cauchy in $\mathcal{F}$. Suppose $F_n \to F$ in $\mathcal{F}$.
Then

\[ \left\| U_n - f \right\| = \lim_{n \to \infty} \left\| U_n - f \right\| = \lim_{n \to \infty} |d_n - f| = 0 \]

so \( f = Uf \). It follows that \( U \) is
in isometry from \( \mathcal{D}_0 \) onto \( L^2(\mathcal{D}_0) \).

Now we return to the claim on p. 77.12

Let \( f \in \mathcal{D}(A) \cap \mathcal{D}_0 \). Suppose \( g \in \mathcal{D}_0^+ \). Then

for \( \varepsilon > 0 \), as \( (1+i\varepsilon A)^{-1} g \in \mathcal{D}(A) \)

\[
\left( \frac{1}{1+i\varepsilon A} g, A f \right) = \left( A^{-1} g, f \right)
\]

Let \( u_n \in \mathcal{D}_0 \), \( u_n \to f \). Then

\[
\left( \frac{1}{1+i\varepsilon A} g, A f \right) = \lim_{n \to \infty} \left( \frac{1}{1+i\varepsilon A} g, A u_n \right)
\]

\[
= \lim_{n \to \infty} \left( g, \frac{1}{1+i\varepsilon A} u_n \right)
\]

But for \( u_n = \sum_{j} a_j \frac{e^j}{A - j} \), \( \frac{1}{1+i\varepsilon A} u_n = \sum_{j} a_j \frac{A}{1+i\varepsilon A - j} e^j \)
\[ e = \sum \frac{a_j}{i} \left( \frac{-i \alpha + 1}{1 - i \alpha} + \frac{1}{1 - i \alpha} \right) \frac{1}{A - i \beta} e_0 \]

\[ = \sum \frac{a_j}{i} \frac{1}{A - i \beta} + \sum \frac{a_j}{i} \frac{1}{1 - i \alpha} \frac{1}{A - i \beta} e_0 \]

which clearly belongs to \( D_0 \), and so \( \varphi = 1 \) to \( y \). Thus

\[ \left( \frac{1}{i(1 + \alpha A - i \beta)} - A \frac{e_0}{e} \right) = 0 \]

and letting \( \varepsilon \to 0 \) we see that \( \Theta_0, A \neq 0 = \varepsilon \Rightarrow \]

A takes \( D(A) \cap \mathbb{H}_0 \) into \( H_0 \). Also\n
\( D(A) \cap \mathbb{H}_0 \) is clearly dense in \( \mathbb{H}_0 \) as \( D_0 \subset D(A) \).

Furthermore \( A_0 = A \beta \) \( D(A) \cap \mathbb{H}_0 \) is clearly symmetric.

Let \[ q = \sum \frac{a_j}{i} \frac{1}{A - i \beta} e_0 \in D_0, \beta \neq i, \text{ and take} \]

\[ q = \frac{1}{A - i} q = \sum \frac{1}{A - i} \frac{a_j}{i} \frac{1}{A - i} e_0 = \frac{1}{(A - i)(A - 3i)} e_0 \]

\[ = \left( \frac{1}{A - i} \frac{1}{A - 3i} \right) \frac{1}{i - 3i} e_0 \]

\( q \in D_0 \subset D(A) \), we have \( (A - i)^2 q = e_0 \), and we
conclude that \( \text{Ran}(A_0 - i^+) \) is dense in \( H_0 \), and

similarly, \( \text{Ran}(A_0 + i^+) \) is dense, and hence \( A_0 \) is e.s. adjoint. But \( A_0 \) is closed.

Indeed, if \( D(A) \cap H_0 \ni f_n \to f \),

and \( A_0 f_n = g \), then

\[
D(A) \ni f_n \to f.
\]

and

\[
A f_n = A_0 f_n - g.
\]

and no \( f \in D(A) \) s.t. \( A f = g \).

But as \( f_n \in H_0 \), \( f \in H_0 \), so \( f \in D(A) \cap H_0 \) and

\[
f \in \text{Dom} A_0 \quad \text{s.t.} \quad A_0 f = A f = g. \quad \text{Thus}
\]

\( A_0 = \overline{A_0} \) is s. adjoint.

Finally, \( A_0 \) is a core for \( A_0 \). Indeed.

The above calculation shows that \( A_0 \cap D_0 \) is a symmetric operator and \( A_0 \cap D_0 \) have

dense ranges. Thus \( \overline{A_0 \cap D_0} \) is s. adjoint.
\[ A_0 \cup B_0 \subset A_0 \] and as s. adj. ops are maximally symmetric, we must have

\[ A_0 = A_0 \cup B_0 \]

so \( B_0 \) is a core for \( A_0 \).

**Lecture 6** Finally we show that

\[(17.20.1) \quad D(A_0) = \{ \phi \in H_0 : \lambda \phi(x) = \lambda \psi(x) \in L^2(\Omega) \} \]

and

\[(17.20.2) \quad (U A_0 U^{-1} \psi)(x) = \lambda \psi(x) \] for \( \psi = U \phi, \phi \in D(A_0) \)

Let \( \psi = \sum \frac{a_j e_0}{\lambda - A - \delta i} \in D_0 \subset D(A_0) \)

Then

\[ A_0 \psi = \sum \frac{a_j}{\lambda - A - \delta i} e_0 \]

\[ = \sum \frac{a_j}{(A - \delta i + \delta i)} \frac{1}{\lambda - A - \delta i} e_0 \]

\[ = (\sum a_j) e_0 + \sum \frac{a_j \delta i}{A - \delta i} e_0 \]

Now

\[ (U - e_0)(x) = \frac{1}{1 + i e \lambda} \]

\[ = U e_0 = 1 \in \ell^1(\Omega) \]