Note that $C_0^\infty(\mathbb{R})$ is a core for T i.e.

if $f \in D(T)$ then $\exists f_n \in C_0^\infty(\mathbb{R})$ s.t.

$$f_n \to f \quad \text{and} \quad i f_n' \to Tf .$$

Lecture 5

Exple 2

Let $\mathcal{H} = \ell^2(-\infty, \infty) = \{ a = \{a_n\}_{-\infty}^{\infty} : \sum_{-\infty}^{\infty} |a_n|^2 < \infty \}$

Let $L : \mathcal{H} \to \mathcal{H}$ s.t. $(La)_n = a_{n+1}$

i.e. L shifts to the left. $L^* = R$ (why?) with $(Ra)_n = a_{n-1}$

= shift to right. Let $A = R + L = L^* + L = A^*$.

which is a bdd s. adj. operator

Now map $\mathcal{H}$ into $L^2[0,1]$ by

$$U : \{a_n\} \to \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$$

Then

$U L U^{-1}$ is mult. by $e^{-2\pi i x}$ in $L^2(0,1)$

and

$U R U^{-1}$ is mult. by $e^{2\pi i x}$ in $L^2(0,1)$

so UAU^{-1} is mult. by $2\cos 2\pi x$

Exercise Construct the necessary transformation needed to represent A is multiplication by x on $L^2(\mathbb{R}, d\mu_1) \oplus L^2(\mathbb{R}, d\mu_2)$: here μ_1 and μ_2 have support in $[-2,2]$. What is the nature of $d\mu_1, d\mu_2$? Any points, etc? A has uniform multiplicity 2.

Exple 3 Consider S_α as in (61.2).

We see that $S_\alpha f = \lambda f \Rightarrow if' = \lambda f$

$\Leftarrow$ $f = c\, e^{-i\lambda x}$ and $f(1) = e^{i\alpha} f(0)$

$\Rightarrow$ $e^{-i\lambda} = e^{i\alpha}$ $\Rightarrow$ $\lambda = \lambda_n = -\alpha + 2n\pi$, $n \in \mathbb{Z}$

Now $f_n = e^{-i\lambda_n x} = e^{i\alpha x} e^{-i2n\pi x}$, $n \in \mathbb{Z}$

is clearly a complete orthonormal set of vectors in $L^2(0,1)$. Indeed $g \perp f_n \Leftrightarrow \int_0^1 \bar{g}(x) e^{i\alpha x} e^{-i2n\pi x} dx$ $\forall n$

$\Rightarrow \bar{g}(x) e^{i\alpha x} = 0 \Rightarrow g(x) = 0$. And (f_n, f_m)

$= \int e^{-i\alpha x} e^{2in\pi x} e^{i\alpha x} e^{-2\pi i m x} dx = \delta_{n,m}$.

Thus the map $\varphi \mapsto \sum_n (f_{n,\alpha}, \varphi) f_{n\alpha}$.

is a unitary map which can then be rewritten (exercise) as giving rise to $U_\alpha : L^2(0,1) \to L^2(\mathbb{R}, d\mu_\alpha)$ where $d\mu_\alpha(\lambda)$ is pp, which turns S_α into multiplication by λ.

Insert 77.1 → 77.

Amongst the most important functions $f(A)$ in the functional calculus for A, are the projection operators mentioned in our first lecture

$$P_\Omega = \chi_\Omega(A)$$

where Ω is a Borel subset of $\mathbb{R}$. As $h \to h(A)$ is a $*$-homomorphism

$$P_\Omega^2 = \chi_\Omega(A)\,\chi_\Omega(A) = \chi_\Omega^2(A) = \chi_\Omega(A) = P_\Omega$$

and $$P_\Omega^* = \overline{\chi}_\Omega(A) = \chi_\Omega(A) = P_\Omega$$

so P_Ω is a self-adj. projection for each Ω.

The fact that $h \to h(A)$ is a $*$-homomorphism

We now prove the spectral Th^m in multiplication operator form (see Th^m 69.1.

Recall Herglotz's Theorem:

Suppose $F(z)$ is an analytic map from $\mathbb{C}_+$ into $\mathbb{C}_+$, then $F(z)$ has a representation

(77.1.1) $$F(z) = az + b + \int_{\mathbb{R}} \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) d\mu(s)$$

for some Borel measure $d\mu(s)$ on $\mathbb{R}$ such that

(77.1.2) $$\int_{\mathbb{R}} \frac{d\mu(s)}{1+s^2} < \infty$$

and for some constants $a \geq 0$ and b.

Proof: Exercise. Note that $\frac{1}{s-z} - \frac{s}{s^2+1} \sim \frac{1}{s^2}$ as $|s| \to \infty$.

Note that if $z = u + iv$, $v > 0$, then

(77.1.3) $$\operatorname{Im} F(u+iv) = av + \operatorname{Im} b + \frac{1}{2i} \left[\int \left(\frac{1}{s-z} - \frac{s}{s^2+1} \right) d\mu(s) - \int \left(\frac{1}{s-\bar{z}} - \frac{s}{s^2+1} \right) d\mu(s) \right]$$

$$= av + \operatorname{Im} b + \frac{1}{2i} \int \left(\frac{1}{s-z} - \frac{1}{s-\bar{z}} \right) d\mu = av + \operatorname{Im} b + v \int \frac{d\mu}{|s-z|^2}$$

Note that if $\operatorname{Im} b \geq 0$, then clearly $\operatorname{Im} F(u+iv) \geq 0$, as it should be. But we note that if

$d\mu(s) = ds$, then

$$v \int \frac{ds}{|s-z|^2} = v \int \frac{ds}{(s-u)^2+v^2} = \int \frac{dt}{1+t^2} = \pi$$

and so even if $\operatorname{Im} b < 0$,

$$\operatorname{Im} b + v \int \frac{d\mu}{|s-z|^2} \geq 0$$

if $d\mu = \gamma\, ds$ with $\gamma\pi > |\operatorname{Im} b|$. We will eventually show that b and a may be taken to be 0. (see (77.4.3) and (77.11.1))

Now for $A = A^*$ set, fix $e_0 \in \mathcal{H}$ $\boxed{\|e_0\|=1}$ and set

$$F_0(z) = \left(e_0, \frac{1}{A-z} e_0\right), \qquad z \in \mathbb{C}_+$$

Then as $\left((A-z)^{-1}\right)^* = (A-\bar{z})^{-1}$,

$$\operatorname{Im} F_0(z) = \frac{1}{2i}\left[\left(e_0, \frac{1}{A-z} e_0\right) - \overline{\left(e_0, \frac{1}{A-z} e_0\right)}\right]$$

$$= \frac{1}{2i}\left(\left(e_0, \frac{1}{A-z} e_0\right) - \left(\frac{1}{A-z} e_0, e_0\right)\right)$$

$$= \frac{1}{2i}\left(\left(e_0, \frac{1}{A-z} e_0\right) - \left(e_0, \frac{1}{A-\bar{z}} e_0\right)\right)$$

$$= \frac{1}{2i}\left(e_0, \frac{1}{A-z} - \frac{1}{A-\bar{z}} e_0\right)$$

$$= (\operatorname{Im} z)\left(e_0, \frac{1}{A-z}\,\frac{1}{A-\bar{z}} e_0\right) = (\operatorname{Im} z)\left\|\frac{1}{A-z} e_0\right\|^2$$

i.e.

(77.3.1) $$\operatorname{Im} F_0(z) = \operatorname{Im} z \left\|\frac{1}{A-z} e_0\right\|^2$$

Thus by Herglotz Th^m, $\exists$ $a \geq 0$, b " and a

Borel meas. $d\mu$, $\int_{\mathbb{R}} \frac{d\mu(s)}{1+s^2} < \infty$, such that

(77.3.2) $$\left(e_0, \frac{1}{A-z} e_0\right) = az + b + \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\mu(s)$$

From (77.1.3) and (77.3.1) we have for $z = u + iv$

(77.4.0) $$\left\| \frac{1}{A-z} e_0 \right\|^2 = a + \frac{\operatorname{Im} b}{v} + \int \frac{d\mu(s)}{(s-u)^2 + v^2}$$

Now for any $f \in D(A)$, $z = u + iv$, $v \neq 0$

$$\|(A-z)f\|^2 = \|(A-u)f\|^2 + v^2\|f\|^2$$

Thus for any $g \in \mathcal{H}$

(77.4.1) $$\|g\|^2 = \left\|(A-u)\frac{1}{A-z} g\right\|^2 + v^2 \left\|\frac{1}{A-z} g\right\|^2$$

$\Rightarrow$

$$\left\| \frac{1}{A-z} g \right\| \leq \frac{1}{|\operatorname{Im} z|} \|g\|$$

$\Rightarrow$

(77.4.2) $$\|(A-z)^{-1}\| \leq \frac{1}{|\operatorname{Im} z|}$$

Letting $v \to +\infty$ in (77.4.0) and using (77.4.1) on the LHS, and monotone convergence on the RHS, we conclude that

(77.4.3) $$a = 0$$

Thus for $z = u + iv$, $v > 0$

(77.5.1) $$v \left\| \frac{1}{A - z} e_0 \right\|^2 = \operatorname{Im} b + v \int \frac{d\mu(s)}{(s-u)^2 + v^2}.$$

Now for $-\infty < \alpha < \beta < \infty$ we have

(77.5.2) $$\int_\alpha^\beta v \left\| \frac{1}{A - u - iv} e_0 \right\|^2 = (\beta - \alpha) \operatorname{Im} b + \int d\mu(s) \int_\alpha^\beta \frac{v\, du}{(u-s)^2 + v^2}$$

$$= (\beta - \alpha) \operatorname{Im} b + \int d\mu(s) \left[\arctan \frac{\beta - s}{v} - \arctan \frac{\alpha - s}{v} \right]$$

Now

$$f(s, v) \equiv \int_\alpha^\beta \frac{v\, du}{(u-s)^2 + v^2} = \int_{\frac{\alpha - s}{v}}^{\frac{\beta - s}{v}} \frac{dt}{1 + t^2}$$

Consider. (i) $s > \beta + 1$

Then $\frac{\alpha - s}{v} < t < \frac{\beta - s}{v} < \frac{-1}{v} < -1$ for $0 < v < 1$

and using the elementary inequality

$$\frac{1}{1 + t^2} \le \frac{2}{(1 - t)^2}$$

$$f(s, v) \le 2 \int_{\frac{\alpha - s}{v}}^{\frac{\beta - s}{v}} \frac{dt}{(1 - t)^2} = 2 \left. \frac{1}{1 - t} \right|_{\frac{\alpha - s}{v}}^{\frac{\beta - s}{v}}$$

$$= 2\left[\frac{1}{1-\frac{\beta-s}{v}} - \frac{1}{1-\frac{\alpha-s}{v}}\right]. = 2\ \frac{1}{1+\frac{s-\beta}{v}}\ \frac{1}{1+\frac{s-\alpha}{v}}.\ \frac{\beta-\alpha}{v}$$

$$= \frac{2v\ (\beta-\alpha)}{(v+(s-\beta))\ (v+s-\alpha)} \le \frac{2\ \beta-\alpha}{(s-\beta)(s-\alpha)}.$$

which is integrable wrt to $q\mu$ on $(\beta+1, \infty)$

(ii) $s < \alpha - 1$.

$$1 < \frac{1}{v'} < \frac{\alpha-s}{v} < t < \frac{\beta-s}{v}$$

and using $\frac{1}{(1+t^2)} \le \frac{2}{(1+t)^2}$

we find

$$f(s,v) \le 2\int_{\frac{\alpha-s}{v}}^{\frac{\beta-s}{v}} \frac{dt}{(1+t)^2} = -\frac{2}{(1+t)}\Bigg|_{\frac{\alpha-s}{v}}^{\frac{\beta-s}{v}}$$

$$= 2\left[\frac{1}{1+\frac{\alpha-s}{v}} - \frac{1}{1+\frac{\beta-s}{v}}\right] = \frac{2(\beta-\alpha)/v}{\left(1+\frac{\alpha-s}{v}\right)\left(1+\frac{\beta-s}{v}\right)}$$

$$= \frac{2v(\beta-\alpha)}{(v+(\alpha-s))(v+\beta-s)} \le \frac{2(\beta-\alpha)}{(\alpha-s)(\beta-s)}$$

which is integrable wrt $q\mu$ on $(-\infty, \alpha-1)$.

(iii) Finally for $\alpha-1 \le s \le \beta+1$

$$|f(s,v)| = \left|\arctan\frac{\beta-s}{v} - \arctan\frac{\alpha-s}{v}\right| \le \pi$$

There is no (77.7)

(77.8)

It follows that for all $s \in \mathbb{R}$ and for all $0 < v < 1$

$$|f(v, s)| \le g(s)$$

for some $g \in L^1(d\mu)$.

As
$$\lim_{v \downarrow 0} \left(\arctan \frac{\beta - s}{v} - \arctan \frac{\alpha - s}{v} \right)$$

$$= 0 \quad \text{if} \quad s > \beta \quad \text{or} \quad s < \alpha,$$

$$= \pi \quad \text{if} \quad \alpha < s < \beta$$

it follows by the dominated convergence thm that

if α and β are not mass points of $d\mu$

(there can only be countable # of such points), then

as $v \downarrow 0$, the RHS of (77.5.4) is given by

$$(\beta - \alpha) \operatorname{Im} b \quad + \pi \mu((\alpha, \beta)).$$

But $\beta - \alpha = \mu_0(\alpha, \beta)$ when μ_0 is Lebesgue measure ds.

As the LHS of (175.2) is positive and depends only on A

(and of course e_0) we see that

$$\tilde{\mu} = \mu + \frac{1}{\pi}(\operatorname{Im} b)\,\mu_0 \geq 0.$$

is a uniquely determined bona fide measure (we use here the fact that α and β are dense and so $\mu([a, b))$

$= \lim_{\varepsilon_n \uparrow 0} \mu(a - \varepsilon_n, b)$ is determined, etc) and

$$\int d\tilde{\mu}(s) \Big/ 1+s^2 < \infty.$$

We have for $z \in \mathbb{C}_+$

$$\int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\tilde{\mu}(s)$$

$$= \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\mu(s) + \frac{1}{\pi}\operatorname{Im} b \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) ds$$

But for $z \in$ $\displaystyle\int \frac{ds}{s-z} - \frac{1}{2}\left(\frac{1}{s+i} + \frac{1}{s-i}\right) ds$

$$= 2\pi i - \frac{1}{2} 2\pi i = \pi i$$

and so

$$\left(e_0, \frac{1}{A-z} e_0\right) = b + \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\mu(s)$$

$$= b + \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\tilde{\mu}(s)$$

$$- i \operatorname{Im} b.$$

and we conclude that $F(z) = (e_0, \frac{1}{A-z} e_0)$

has a Herglotz representation of the form

(77.10.1) $$(e_0, \frac{1}{A-z} e_0) = b + \int \left(\frac{1}{s-z} - \frac{s}{s^2+1}\right) d\mu(s)$$

where $b \in \mathbb{R}$ and $d\mu$ is a measure with

$$\int \frac{d\mu(s)}{1+s^2} < \infty .$$

Returning to (77.5.1) we have for $z = u+iv$, $v \in \mathbb{C}_+$

$$v^2 \left\| \frac{1}{A-z} e_0 \right\|^2 = \int \frac{v^2}{(s-u)^2+v^2} \, d\mu(s).$$

But the LHS is bded by (77.4.2) and

hence by monotone convergence, we conclude that

(77.10.2) $$\int d\mu(s) < \infty .$$

It follows that we can separate the integral

in (77.10.1) to obtain

(77.10.3) $$(e_0, \frac{1}{A-z} e_0) = \hat{b} + \int \frac{1}{s-z} \, d\mu(s)$$

where $\hat{b} = b - \int \frac{s}{s^2+1} \, d\mu$

But we can now let $v \to +\infty$, $z = u+iv$

to conclude that $\hat{b} = 0$. Thus finally

we have the Herglotz representation

(77.11.1) $$\left(e_0, \frac{1}{A-z} e_0\right) = \int \frac{d\mu(s)}{s-z}, \quad z \in \mathbb{C}_+$$

where $\int d\mu(s) < \infty$.

Observe that if $z \in \mathbb{C}_-$, then $\bar{z} \in \mathbb{C}_+$

and so

$$\left(e_0, \frac{1}{A-z} e_v\right) = \overline{\left(\frac{1}{A-z} e_0, e_0\right)}$$

$$= \overline{\left(e_0, \frac{1}{A-\bar{z}} e_0\right.}$$

$$= \overline{\int \frac{d\mu(s)}{s-\bar{z}}}$$

$$= \int \frac{d\mu(s)}{s-z}$$

so (77.11.1) holds for all $z \in \mathbb{C} \setminus \mathbb{R}$.

Now let $\mathcal{D}_0$ be the linear space

$$\mathcal{D}_0 = \left\{ \sum_{i=1}^{n} \frac{a_i}{A - z_i} e_0 : \begin{array}{l} a_i \in \mathbb{C} \\ z_i \in \mathbb{C} \setminus \mathbb{R} \\ z_i \neq z_j \text{ for } i \neq j \\ 1 \leq n < \infty \end{array} \right\}$$

and let $\mathcal{H}_0 = \overline{\mathcal{D}_0}$, the closure of $\mathcal{D}_0$.

We will prove later that:

<u>Claim</u>: A_0, the restriction of A to $D(A) \cap \mathcal{H}_0$, is a self-adjoint operator in $\mathcal{H}_0$ and $\mathcal{D}_0$ is a core for A.

Note first that $e_0 \in \mathcal{H}_0$: $\mathcal{H}_0$ is called the <u>cyclic subspace</u> generated by e_0 and A. Indeed, for any $\varepsilon > 0$

$$\varphi_\varepsilon = \frac{1}{1 + i\varepsilon A} e_0 \in \mathcal{D}_0$$

Let $\eta > 0$ be given: we show that for $\varepsilon > 0$ suff. small, $\|\varphi_\varepsilon - e_0\| < \eta$. Now as the domain of A is dense, $\exists f \in D(A)$ s.t. $\|f - e_0\| < \eta/3$

Then

$$\varphi_\varepsilon - e_0 = \frac{1}{1+i\varepsilon A} f + \frac{1}{1+i\varepsilon A}(e_0 - f) - e_0$$

$$= f - \frac{1}{1+i\varepsilon A} i\varepsilon A f + \frac{1}{1+i\varepsilon A}(e_0 - f)$$

$$- e_0$$

Thus, using the fact that $\|\frac{1}{1+i\varepsilon A}\| \le 1$ which follows from (77.4.2)

$$\|\varphi_\varepsilon - e_0\| \le \|f - e_0\| + \varepsilon \|Af\| + \|e_0 - f\|$$

$$\le \frac{2}{3}\eta + \varepsilon \|Af\|$$

Choosing $\varepsilon > 0$ s.t. $\varepsilon\|Af\| < \eta/3$, we are done.

What we are in fact shown is that

$$\varepsilon \mapsto \frac{1}{1+i\varepsilon A} f \quad \text{is}$$

continuous ~~for~~ all f. As $\|\frac{1}{1+i\varepsilon A} e_0\|^2 = \int \frac{d\mu(s)}{|1+i\varepsilon s|^2}$ by (77.5.1) for $b=0$, $z = -\frac{1}{i\varepsilon}$, we conclude the $\boxed{\int d\mu = \|e_0\|^2 = 1}$

Let $\varphi = \sum_{i=1}^{n} \frac{a_i}{A - z_i} e_0 \in \mathcal{D}_0$

Then $\|\varphi\|^2 = \sum_{i,j} \bar{a}_i a_j \left(\frac{1}{A - z_i} e_0, \frac{1}{A - z_j} e_0\right)$

$$= \sum_{i,j} \bar{a}_i a_j \left(e_0, \frac{1}{A - \bar{z}_i} \frac{1}{A - z_j} e_0\right)$$

Now if $\bar{z}_i \neq z_j$

$$\left(e_0, \frac{1}{A-\bar{z}_i}\frac{1}{A-z_j} e_0\right)$$

$$= \left(e_0, \frac{1}{A-\bar{z}_i} - \frac{1}{A-z_j} e_0\right)\frac{1}{\bar{z}_i - z_j}$$

$$= \int d\mu(s)\left(\frac{1}{s-\bar{z}_i} - \frac{1}{s-z_j}\right)\frac{1}{\bar{z}_i - z_j}$$

(77.14.1) $$= \int d\mu(s)\frac{1}{s-\bar{z}_i}\frac{1}{s-z_j}$$

Also if $\bar{z}_i = z_j$, then

$$\left(e_0, \frac{1}{A-\bar{z}_i}\frac{1}{A-z_j} e_0\right)$$

$$= \left(e_0, \frac{1}{(A-z_j)^2} e_0\right) = \frac{d}{dz}\Big|_{z=z_j}\left(e_0, \frac{1}{A-z} e_0\right)$$

$$= \frac{d}{dz}\Big|_{z_j}\int d\mu(s)\frac{1}{s-z}$$

$$= \int d\mu(s)\frac{1}{(s-z_j)^2} = \int\frac{d\mu(s)}{(s-\bar{z}_i)(s-z_j)}.$$

so in both situations, (77.14.1) holds.

Have,

$$\|u\|^2 = \sum_{i,j}\bar{a}_i a_j \int d\mu\frac{1}{s-\bar{z}_i}\frac{1}{s-z_j}$$

$$= \int d\mu(s) \, |q(s)|^2$$

where $q(s) = \sum_{i=1}^{n} \frac{a_i}{s - z_i}$

ie

(17.15.1) $$\left\| \sum_i^n \frac{a_i}{A - z_i} e_0 \right\|^2 = \int d\mu(s) \, |q(s)|^2 .$$

Suppose φ has 2 representations

$$\varphi = \sum_i^n \frac{a_i}{A - z_i} e_0 = \sum_i^{\tilde{n}} \frac{\tilde{a}_i}{A - \tilde{z}_i} e_0$$

Wlog we can assume $n = \tilde{n}$ and $z_i = \tilde{z}_i$

Then $0 = \left\| \sum \frac{a_i - \tilde{a}_i}{A - z_i} e_0 \right\|^2 = \int \left| \sum_{i=1}^{n} \frac{a_i - \tilde{a}_i}{s - z_i} \right|^2 d\mu(s)$

ie $\sum \frac{a_i}{s - z_i} = \sum \frac{\tilde{a}_i}{s - \tilde{z}_i}$ $d\mu$-a.e.

Thus the map

$$\mathcal{D}_0 \ni \varphi = \sum_{i=1}^{n} \frac{a_i}{A - z_i} e_0 \mapsto \sum \frac{a_i}{s - z_i} \in L^2(d\mu)$$

is well-defined and is an isometry by (17.15.1)

and hence extends to an isometry U from $\mathcal{H}_0$

into $L^2(d\mu)$.

Let $f \in L^2(d\mu)$. Then a standard argument shows that $\exists$ continuous function $f_n(s)$ with compact support : $\|f_n - f\|_{L^2(d\mu)} \to 0$ $\boxed{= \{\sum a_i/_{s-z_i}\}}$

But by Stone-Weierstrass, $U(D_0)|_\Lambda$ is L^∞ dense in the class of continuous functions that decay at ∞.

Hence $\exists\ \varphi_n \in U(D_0)$ s.t. $\|\varphi_n - f_n\|_{L^\infty} \leq \frac{1}{n}$. But then as $\int d\mu = 1$

$$\|f_n - \varphi_n\|^2 = \int |f_n - f_n|^2 d\mu(s) \leq \frac{1}{n} \int d\mu(s) = \frac{1}{n} \to 0$$

Then $\|\varphi_n - f\|_{L^2(d\mu)} \to 0$ as $n \to \infty$.

Let $\psi_n = \sum \frac{a_i^{(n)}}{A - z_i^{(n)}} e_0$ where

$\varphi_n = \sum \frac{a_i^{(n)}}{s - z_i^{(n)}}$. Then $\|\psi_n\|_{\mathcal{H}} = \|\varphi_n\|_{L^2}$

And so as $\{\varphi_n\}$ is Cauchy in L^2, $\{\psi_n\}$ is Cauchy in $\mathcal{H}$. Suppose $\psi_n \to \psi \in \mathcal{H}_0$

Then

$$\|U\varphi - f\| = \lim_n \|U\varphi_n - f\|$$

$$= \lim_n \|\phi_n - f\|$$

$$= 0$$

so $f = U\phi$. It follows that U is an isometry from H_0 onto $L^2(q\mu)$.

Now we return to the claim on p 17.12

Let $f \in D(A) \cap H_0$. Suppose $g \in H_0^{\perp}$. Then for $\varepsilon > 0$, as $(1+i\varepsilon A)^{-1} g \in D(A)$

$$\left(\frac{1}{1+i\varepsilon A} g, Af\right) = \left(A \frac{1}{1+i\varepsilon A} g, f\right)$$

Let $\varphi_n \in D_0$, $\varphi_n \to f$. Then.

$$\left(\frac{1}{1+i\varepsilon A} g, Af\right) = \lim_{n\to\infty} \left(A \frac{1}{1+i\varepsilon A} g, \varphi_n\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+i\varepsilon A} g, A\varphi_n\right)$$

$$= \lim_{n\to\infty} \left(g, \frac{1}{1-i\varepsilon A} A\varphi_n\right)$$

But for $\varphi_n = \sum_i \frac{a_i}{A - z_i} e_0$, $\frac{1}{1-i\varepsilon A} A\varphi_n = \sum a_j \frac{A}{1-i\varepsilon A} \frac{1}{A - z_j} e_0$

$$= \sum \frac{a_j}{-i\varepsilon}\left(\frac{-i\varepsilon A - 1}{1 - i\varepsilon A} + \frac{1}{1-i\varepsilon A}\right)\frac{1}{A - z_j} e_0$$

$$= \sum \frac{a_j}{-i\varepsilon}\frac{1}{A - z_j} e_0 + \sum \frac{a_j}{-i\varepsilon}\frac{1}{1-i\varepsilon A}\frac{1}{A-z_j} e_0$$

which clearly belongs to D_0, and so is $\perp$ to g. Thus

$$\left(\frac{1}{1+i\varepsilon A} g, Af\right) = 0$$

and letting $\varepsilon \downarrow 0$ we see that $(g, Af) = 0$ so A takes $D(A) \cap H_0$ into H_0. Also $D(A) \cap H_0$ is clearly dense in H_0 as $D_0 \subset D(A)$. Furthermore $A_0 \equiv A \upharpoonright D(A) \cap H_0$ is clearly symmetric.

Let $\varphi = \sum_j \frac{a_j}{A - z_j} e_0 \in D_0$, $z_j \neq i$, and set

$\psi = \frac{1}{A-i}\varphi = \sum_j \frac{1}{A-i}\frac{a_j}{A-z_j} e_0$. As $\left(\frac{1}{A-i}\frac{1}{A-z_j}\right) e_0$

$$= \left(\frac{1}{A-i} - \frac{1}{A-z_j}\right)\frac{1}{i - z_j} e_0 \in D_0,$$ we see that

$\psi \in D_0 \subset D(A)$, we have $(A-i)\psi = \varphi$, and we conclude that $\operatorname{Ran}(A_0 - i)$ is dense in $\mathcal{H}_0$, and similarly $\operatorname{Ran}(A_0 + i)$ is dense, and hence A_0 is e.s.adj. But A_0 is closed.

Indeed, if $D(A) \cap \mathcal{H}_0 \ni f_n \to f$.

and $A_0 f_n \to g$, then

$$D(A) \ni f_n \to f.$$

and

$$A f_n = A_0 f_n \to g.$$

and so $f \in D(A)$ & $Af = g$.

But as $f_n \in \mathcal{H}_0$, $f \in \mathcal{H}_0$. & so $f \in D(A) \cap \mathcal{H}_0$ ie

$f \in \operatorname{Dom} A_0$ & $A_0 f = Af = g$. Thus

$A_0 = \overline{A_0}$ is s.adj.

Finally $\mathcal{D}_0$ is a core for A_0. Indeed.

the above calculation shows that $A_0 \restriction \mathcal{D}_0$ is a symmetric operator and $A_0 \restriction \mathcal{D}_0 \pm i$ have dense ranges. Thus $\overline{A_0 \restriction \mathcal{D}_0}$ is s.adj. But

$\overline{A_0 \upharpoonright D_0} \subset \overline{A_0} = A_0$ and as s. adj. ops

are maximally symmetric, we must have

$$A_0 = \overline{A_0 \upharpoonright D_0}$$

$\therefore$ D_0 is a core for A_0

Lecture 6 Finally we show That

(17.20.1) $$D(A_0) = \{ \varphi \in \mathcal{H}_0 : \lambda U\varphi(\lambda) = \lambda \psi(\lambda) \in L^2(d\mu) \}$$

and

(17.20.2) $$(U A_0 U^{-1} \psi)(\lambda) = \lambda \psi(\lambda) \quad \text{for } \psi = U\varphi, \ \varphi \in D(A_0)$$

Let $\varphi = \sum_j \frac{a_j}{A - z_j} e_0 \in D_0 \subset D(A_0)$

Then $A_0 \varphi = \sum a_j A \frac{1}{A - z_j} e_0$

$$= \sum a_j (A - z_j + z_j) \frac{1}{A - z_j} e_0$$

$$= (\sum a_j) e_0 + \sum a_j z_j \frac{1}{A - z_j} e_0$$

Now $\left(U \frac{1}{1 + i\varepsilon A} e_0 \right)(\lambda) = \frac{1}{1 + i\varepsilon\lambda}$

$\underset{\varepsilon \downarrow 0}{\Rightarrow}$ $U e_0 = 1 \in L^2(d\mu)$