We conclude from the above that if

\[ T^* = T', \quad D(T) = \mathcal{D} \]

then \[ D(T) = \{ \mathbf{f} \in \mathcal{L}^2 ; \mathbf{f} \to \mathbf{f}' \mathbf{e} \mathcal{L}^2 \} \]

\[ T^* \mathbf{f} = \mathbf{f}' \]

---

**Lecture 3**

**Definition 35.1: Adjoint Operator** \( T^* \)

Let \( T \) be a densely defined operator in \( \mathcal{D} \). Let

\[ D(T^*) \] is the set of \( \mathbf{f} \in \mathcal{D} \) for which \( T \mathbf{f} \in \mathcal{D} \)

\( s^+ \)

\[ (35.2) \quad (T \mathbf{u}, \mathbf{e}) = (\mathbf{u}, \mathbf{m}) \quad \forall \mathbf{u} \in D(T) \]

Then for \( \mathbf{u} \in D(T^*) \),

\[ T^* \mathbf{u} = \mathbf{m} \]

(35.3)

Note: That as \( T \) is densely defined, \( \mathbf{m} \) is unique in (35.2) and hence \( T^* \) is well-defined.

Note: \( \mathbf{u} \in D(T^*) \) if and only if \( (T \mathbf{u}, \mathbf{e}) = (\mathbf{u}, \mathbf{m}) \) is a bounded linear functional on \( \mathcal{L}^2 \).
Note:

\[(36.1) \quad \zeta \in D(T^*) \iff \left( (T^* \zeta, \Phi) \leq C \| \Phi \|_{L^1} + 4 \in D(T) \right) \quad \text{(why ?)}\]

\[(36.2) \quad S \subset T = T^* C S^*.\]

Clearly, \(0 \in D(T^*)\), but unlike the case of bounded operators, \(D(T^*)\) may not be dense. In fact,

there are examples where \(D(T^*) = \{0\}\).

\[\text{Example 36.3} \quad \text{bounded}\]

Let \(\zeta\) be a measurable function such that

\[\zeta \in L^2(\mathbb{R}) \quad \text{and} \quad D(T) = \{ \zeta \in L^2(\mathbb{R}) : \int \left| \zeta(x) \right|^2 \, dx < \infty \} \]

Let \(\Phi_0 \equiv 0\) be some fixed vector in \(L^2(\mathbb{R})\). Define

\[T \Phi = (\zeta, \Phi) \Phi_0 \]

\[= \left( \int \zeta(x) \Phi(x) \, dx \right) \Phi_0, \quad \Phi \in D(T).\]

As \(D(T)\) certainly contains all \(L^2\) functions with compact support, we see that \(T\) is densely defined.
Now suppose \( f \in D(T^*) \). Then for some \( m \)

\[
(\mathbf{r}^*, f) = \mathbf{r}^* f \in L^2
\]

\[
(f, \mathbf{r}^* f) = (f, \mathbf{r}) = 0 \forall \mathbf{r} \in D(T)
\]

\[
\iint f(x, \mathbf{r}) \overline{\psi(x, dx)} \mathbf{r}(x) dx = 0.
\]

As \( \mathbf{r} \) can be any \( L^1 \) func. with compact supp we must show

\[
\eta(x) = (f_0, \mathbf{r}) \mathbf{r}(x)
\]

But R.H.S \( \in L^2 \) \( \Rightarrow \) \( (f_0, \mathbf{r}) = 0 \). Thus \( D(T^*) \) is L

\[
to f_0 \text{ and hence cannot be dense}. \text{ Note also that}
\]

\[
+^* f = \eta = 0 \text{ for } f \in D(T^*).
\]

**Exercise:** Extend the above construction to show

\[
T \text{ densely defined } T^* D(T^*) \in \mathcal{B}(L^2)
\]

Observe that \( T \) in 36.3 is not closable.

Indeed as \( f \in L^2 \), \( T \) compact subsets \( S_n \) of \( L^2 \)
such that
\[ \int_{\mathbb{R}} |f(x)|^2 \, dx = n. \]

Let \( y_n = \frac{1}{n} x_{5n} \), \( n \geq 1 \). Clearly \( y_n \in D(T) \) and
\[ \|y_n\| = \frac{1}{n^2} / \|x_{5n}\| = \frac{1}{n^2} \rightarrow 0 \text{ as } n \rightarrow \infty. \]
But \( Ty_n = \langle f, y_n \rangle y_0 = \left( \frac{1}{n} / \|f\| \right) y_0 = y_0 \). So \( y_n \rightarrow 0 \) but \( Ty_n \neq 0 \).

It turns out that there is an intimate relationship between the closability of an operator and the dense definition of its adjoint.

Note that if \( T^* \) is densely defined, then we may define \( T^{**} = (T^*)^* \).

**Theorem 38.1** Let \( T \) be a densely defined operator in \( H \). Then
(a) \( T^* \) is closed
(b) \( T \) is closable \( \iff \) \( T^* \) is densely defined, in which case \( \overline{T} = T^{**} \)
(c) If \( T \) is closable, then \( (T^*)^* = T^* \).
Proof (a) Suppose \( D(T^*) \ni \phi_n \rightharpoonup \phi \rightharpoonup \phi \).

Now \( \forall \psi \in D(T) \), \( (T\psi, \phi_n) = (\psi, T^*\phi_n) \).

Letting \( n \to \infty \) we obtain \( (T\psi, \phi) = (\psi, T^*\phi) \) \( \forall \psi \in D(T) \).

Hence \( \phi \in D(T^*) \) and \( T^*\phi = \phi \). Thus \( T^* \) is closed.

(b) Suppose \( D(T^*) \) is dense and \( \phi \in D(T^*) \). Then

\[
(T\psi, \phi) = (\psi, T^*\phi) \quad \forall \psi \in D(T) \tag{39.1}
\]

Now suppose \( D(T) \ni \phi_n \rightharpoonup \phi \rightharpoonup \phi \).

Then \( T\phi_n \rightharpoonup T\phi \).

Then from (39.1), \( (\psi, \phi) = (0, T^*\phi) = 0 \).

But as \( D(T^*) \) is dense, we conclude \( \phi = 0 \). Thus \( T \) is closable.

The proof of the converse is more involved. Let

\[
T^*(T) = \text{graph of } T = \{ (x, T\phi) : x \in D(T) \} \subset H \times H.
\]

\( H \times H \) is a Hilbert space with inner product \( \langle x, y, x', y' \rangle \)

\( \langle x, y, x', y' \rangle = \langle x, x' \rangle + \langle y, y' \rangle \).
Now

\[ \langle \phi, g \rangle \perp T^*(T) \text{ in \# \times \#} \]

\[ \iff (\phi, u) + (g, Ty) = 0 \quad \forall u \in D(T) \]

\[ \iff (y, Ty) = (-\phi, u) \quad \forall u \in D(T) \]

\[ \iff g \in D(T^*) \text{ and } \tau^x y = -\phi \]

(40.1) Hence,

\[ T(T) = \{ \langle T^*g, y \rangle : g \in D(T^*) \} \]

Now (exercise):

\[ T \text{ is closable } \iff \overline{T(T)} \text{ is a graph and } \overline{T(T)} = T(T) \]

Clearly, \( T \text{ is closed } \iff T(T) \text{ is closed} \)

Thus if \( T \text{ is closable, then from (40.1) } \)

\[ \# \times \# = \overline{T(T)} \oplus (\overline{T(T)})^\perp = T(T) \oplus (T(T))^\perp. \]

(Here we use \((T(T))^\perp = (T(T))^\perp \).

Thus any \( \langle \phi, g \rangle \in \# \times \# \) can be written

uniquely as

\[ \langle \phi, g \rangle = \langle \phi, Ty \rangle + \langle -T^*\phi, \phi \rangle, \quad \forall \phi \in D(T), \]

\[ \phi \in D(T^*) \]

If \( D(T^*) \) is not dense, then \( \exists \, \psi \neq 0, \psi \perp D(T^*) \).

Then

\[ \langle \psi, \phi \rangle = \langle \phi_0, Ty \rangle + \langle -T^*\phi_0, \phi_0 \rangle \text{ for suitable } \phi_0 \]
\[ \Phi_0 \in D(T^*), \quad \Phi_0 \in D(T)^* \]

i.e.

\[ (41.1) \quad 0 = \Phi_0 - T^* \Phi_0 \]
\[ (41.2) \quad \Phi = T^* \Phi_0 + \Phi_0 \]

As \((\Phi, \Phi_0) = 0\), we have from \((41.2)\)

\[ \| \Phi_0 \|_T + (T^* \Phi_0, \Phi_0) = 0 \]

but by part (c) (see below), \(T^* = T^*\), and so

\[ \| \Phi_0 \|_T + (\Phi_0, T^* \Phi_0) = 0 \]

by \((41.1)\) we then obtain

\[ \| \Phi_0 \|_T + \| \Phi_0 \|_T = 0 \]
\[ \therefore \Phi_0 = \Phi_0 = 0 \quad \Rightarrow \quad \Phi = 0 \], which is a contradiction.

Thus \(T^*\) is densely defined.

\[ T(T) = T(T) \text{ is closed.} \]

For \(T\) closable, and from \((40.1)\) (recall \(X = X^{**}\) for any closed set \(X \subset T\)).

\[ T^* (T) = T (T^{**}) \]
\[ = \left\{ g \in T^* : g \in D(T^*) \right\} \]

but by the proof of \((40.1)\), this is just \(T^* (T^{**})\).

Thus \(T = T^{**}\).
Finally we prove (c). Suppose $T$ is closable.

We have that $T^* \subseteq \text{Dom}(T^*) \implies (\mathcal{L}^* + (4, \mathcal{L}^* \mathcal{L}^*) + 4) \subseteq T^*$.

But then closable $T$, we obtain $(\mathcal{L}^* + 4) \subseteq (4, \mathcal{L}^* \mathcal{L}^*) + 4$.

Since $\mathcal{L}^* \subseteq \text{Dom}(T)$, hence $T^* \subseteq (\mathcal{L}^* \mathcal{L}^*)^*$. Conversely, as $\mathcal{T} \subseteq T$, we always have $(\mathcal{L}^* \mathcal{L}^*)^* \subseteq T^* \subseteq (\mathcal{L}^* \mathcal{L}^*)^*$.

---

**Definition 4.2.1 (spectrum & resolvent set)**

Let $T$ be a (densely defined) closed operator in $\mathbb{H}$.

A complex number $\lambda$ is in the resolvent set $\rho(T)$ of $T$ if $\lambda - T$ is a bijection of $\text{Dom}(T)$ onto $\mathbb{H}$.

If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda - T)^{-1} : \mathbb{H} \to \text{Dom}(T)$ is called the resolvent of $T$ at $\lambda$. Here $R_\lambda(T) = 1_{\text{Dom}(T)}$.

Exercise: If $\lambda - T$ is closed, then $\lambda - T$ is closed on $\text{Dom}(T)$ and $\lambda \in \rho(T)$.

A very important fact is that if $\lambda \in \rho(T)$, then $R_\lambda(T)$ is a bounded operator. This fact lies at the heart of the analytic viability of closed operators.
As \( R_x(T) \) is everywhere defined, it is sufficient by the Closed Graph Theorem to show that \( f \)

\[ f_n \to f \quad \text{and} \quad R_x(T) f_n \to g \]

Then \( R_x(T) f = g \). But \( R_x(T) f_n \to \text{dom}(T) \)

\[ = D(\lambda - T) \]

and \( (\lambda - T)(R_x(T) f_n) = f_n \to f \)

Also \( R_x(T) f_n \to g \)

Hence as \( \lambda - T \) is closed on \( D(T) \), \( g \in D(\lambda - T) \)

\[ (\lambda - T)g = f \quad \text{iff} \quad g = R(T)f , \] which is what we wanted to prove.

**Thm 43.1**

Let \( T \) be closed in \( \mathfrak{B} \). Then \( P(T) \) is an open subset of \( C \) on which \( R_x(T) \) is an analytic operator valued function. Furthermore
\{ R_\lambda(T) : \lambda \in \rho(T) \}

is a commuting family of bounded operators satisfying

\[(44.1) \quad R_\lambda(T) - R_\mu(T) = (\mu - \lambda) R_\mu(T) R_\lambda(T) \]  

**Proof:** Exercise.

We define

\[ \sigma(T) = \text{spectrum of } T = \mathbb{C} \setminus \rho(T) \]

Clearly \( \sigma(T) \) is a closed subset of \( \mathbb{C} \). The spectrum \( \sigma(T) \) can be broken up in different ways into point spectrum, etc., as mentioned in Lecture 1.

**Exercise:** Let \( \Omega \) be a closed set in \( \mathbb{C} \). Show that \( \Omega \) is the spectrum of some operator in \( L^2(\mathbb{R}^d) \) (Hint: Show that \( \Omega \) has a countable dense subset.)

The spectrum of an operator is a subtle matter, as we see from the following 2 examples:

Let \( AC(0,1) = \{ f : f \in L^1(0,1) \} : f \) is absolutely continuous on \([0,1]\) and \( f'(x) \in L^1(0,1) \).

Let

\[ D(T_1) = \{ \psi : \psi \in AC[0,1] \} \]

\[ D(T_2) = \{ \psi : \psi \in AC[0,1], \psi(0) = 0 \} \]

And let

\[ T_j \psi = i \psi'(x) \quad \text{if } \psi \in D(T_j), \quad j = 1, 2. \]
A similar calculation to that given above shows that both $T_1$ and $T_2$ are closed operators.

However

\[(4.5.1) \quad \sigma(T_1) = \mathbb{C} \]

\[(4.5.2) \quad \sigma(T_2) = \emptyset \]

Proof: To see that $\sigma(T_1) = \mathbb{C}$ simply observe that $\lambda = e^{ix} \in D(T_1)$ for any $\lambda \in \mathbb{C}$ as

\[(\lambda - T_1)e^{ix} = (\lambda + i\alpha \lambda)e^{-ix} = 0\]

so that $\lambda - T_1$ is not $1-1$ for any $\lambda$. As for $T_2$, suppose that $h \in C$ and $g \in L^2[0, 1]$ are given and we try to solve the equation

\[(4.5.3) \quad (\lambda - T_2)\varphi = g\]

for $\varphi \in D(T_2)$. Then necessarily $\lambda \alpha - i\varphi = g$

i.e. $-i \frac{d}{dx}(e^{ix}\varphi) = g$
\[-i \ e^{ix} f(x) = -i \ e^{ix} f(x) \ \bigg|_{x=0} + \int_0^x e^{i\lambda (x-t)} \ g(t) \ dt \]

\[= \int_0^x e^{i\lambda x} g(t) \ dt \]

\[= \left( S_\lambda g \right)(x). \]

So we take (46.1) as our starting point. Observe that given \(\lambda, g, f = S_\lambda g(x) \in AC(0,1)\) and \(f(0) = 0\), so \(f \in \mathcal{D}(T_2)\). Also

\[\left( \lambda - T_2 \right) f = \left( \lambda - i \ \frac{d}{dx} \right) \ \int_0^x e^{i\lambda (t-x)} \ g(t) \ dt \]

\[= \ i \ \int_0^x e^{i\lambda (t-x)} \ g(t) \ dt \]

\[+ e^{i\lambda (x-x)} \ g(x) \ - i \ \int_0^x e^{i\lambda (t-x)} \ g(t) \ dt \]

\[= g(x) \]

Thus \(\lambda - T_2\) is onto. Also for \(f \in \mathcal{D}(T_2)\)

and \(\left( \lambda - T_2 \right) f = 0\), then as above \(-i \ \frac{d}{dx} (e^{i\lambda x} f) = 0\)

and no \(e^{i\lambda x} f(x) = \text{const} = c\). But \(f(0) = 0\) and \(c = 0\). Thus \(f(x) = 0\) and no \(T_2\) is \(1-1\). It follows
that \( p(t) = C \).

Examples (45.1) (45.2) are in sharp contrast with all situations for bounded operators and self-adjoint operators (see below).

Firstly, if \( T \) is bounded then (exercise),

\( \sigma(T) \) is a bounded, non-empty set. Secondly, if \( T \) is self-adjoint (bounded or unbounded) then

\( \phi \neq \lambda \sigma(T) \subset \mathbb{R} \).

Remark:

By general principles noted above, \( S_1 \) is dense from \( L^1 \to L^1 \); it is of interest to check this directly (exercise!)

We now formally distinguish between

symmetric operators and self-adjoint operators

Defn: A densely defined operator \( T \) an \( \mathcal{H} \) is called
symmetric (or self-adjoint) if

\[(48.1) \quad \Gamma \subseteq \Gamma^*
\]

i.e. if \( \mathcal{D}(\Gamma) \subseteq \mathcal{D}(\Gamma^*) \) and \( \Gamma \Psi = \Gamma^* \Psi \neq 0 \in \mathcal{D}(\Gamma) \)

Equivalently

\[(48.2) \quad (\Gamma \Psi, \Psi) = (\Psi, \Gamma \Psi) \quad \forall \Psi, \Psi \in \mathcal{D}(\Gamma).
\]

Definition:

\( \Gamma \) is called self-adjoint if \( \Gamma = \Gamma^* \) i.e.

\( \Gamma \) is symmetric and \( \mathcal{D}(\Gamma) = \mathcal{D}(\Gamma^*) \)

Clearly if \( \Gamma \) is self-adjoint, then \( \Gamma \) is closed.

Remarks:

\[(48.5) \quad \text{If } \Gamma \text{ is symmetric and bounded, then } \Gamma \text{ is s. adj.}
\]

\[(48.6) \quad \text{If } \Gamma \text{ is symmetric, then by } (48.1) \ \mathcal{D}(\Gamma^*)
\]

is dense. Hence \( \Gamma \) is automatically closable.

As \( \Gamma^* \) is always closed by Theorem 38.1(a), then

If \( \Gamma \) is symmetric, \( \Gamma^* \) is a closed extension
of $T^*$ and we have

$$T \subset T^* = T^{**} \subset T^*$$

In particular for closed sym. operators

$$T \subset T^{**} \subset T^*$$

and for s.a.d. operators

$$T = T^{**} = T^*$$

Thus

$$T = T^{**} = T^*$$

a closed sym. op $T$ is s.a.d. if and only if $T^*$ is symmetric

**Example:** Consider

$$T_3 f = f'(x)$$

with domain $\mathcal{D}(T_3) = \{ f \in L^2(0,1) : f \in AC[0,1], f(0) = f(1), f(0) = f(1) \} = 0 \}$$

As before, we see that $T_3$ is a closed operator.

It is also symmetric: Indeed for $f, g \in \mathcal{D}(T_3)$,
\[(\phi, T_3 g) = \int_0^1 \phi' \, g' = \int_0^1 (\phi', g) = (T_3 \phi, g)\]

Here we use integration by parts which is valid because \( AC [0,1] \) is an algebra (check this).

and \( \phi' \) and \( g' \) vanish at 0 and at 1, so there are no boundary terms.

Claim:

Let

\[D(s) = \{ \phi \in L^1 : \phi \in AC(0,1) \} \]

\[S \phi = i \phi', \ \phi \in D(s)\]

Then \( S = T_3 \).

Indeed for \( \phi \in D(s) \) and \( g \in D(T_3) \), then

\[(50.1) \quad (\phi, T_3 g) = \int_0^1 \phi' \, g' = (i \phi', g) = (S \phi, g)\]

Again use eq. by parts in valid case as \( f(x) \) is cont. on \([0,1]\) (why?) and \( g(T_3 \phi) = g(\phi) = 0\), again there are no boundary terms. Thus by (50.1)
\( \mathcal{P} \in D(T_3^{-1}) \text{ and } T_3^{-1} \mathcal{P} = \mathcal{S} \mathcal{P} \implies \mathcal{S} \subseteq T_3^{-1} \mathcal{P} \)

(Conversely, suppose \( \mathcal{P} \in D(T_3^{-1}) \text{ and } \mathcal{G} \in D(T_3) \).

Then \( (\mathcal{P}, T_3 \mathcal{G}) = \int_0^1 \mathcal{F} \mathcal{G}' = \int_0^1 h \mathcal{G} \) for some \( h \in L^2 \).

\[ h = T_3^{-1} \mathcal{P}. \]

For \( H(x) = \int_0^x h(t) \ dt \) at \( a \), we obtain

as on p.33 that (the units of measure drop out as \( q(0) = q(1) = 0 \))

\[ \int_0^1 \mathcal{F} \mathcal{G}' = -\int_0^1 H(x) \mathcal{G}'(x) \]

and no \( \int_0^1 \mathcal{F} - H(x) \mathcal{G}'(x) \ dx \), \( \forall \mathcal{G} \in D(T_3) \)

and we conclude as before that

\[ \mathcal{F}(x) = H(x) + c = \int_0^x h(t) \ dt + c. \]

Thus \( \mathcal{F} \in AC(0,1) \) and \( \mathcal{F}' = h = T_3^{-1} \mathcal{P} \).

This shows that \( T_3^{-1} \subseteq S \), and hence establishes the claim.

Thus \( T_3^{-1} \mathcal{P} = \mathcal{S} = T_1 \mathcal{P} \).

(See p.44.)