Lecture 15

A key question in mathematics is the stability of mathematical properties under perturbations. Here we are concerned with the following question:

If \( A \) is s.a.d.j. (or e.s.a.) in \( \mathcal{H} \) and \( B \) is a symmetric operator, is \( A + B \) s.a.d.j. (or e.s.a.)? The key result here is due to Kato and Rellich.

**Definition 196.1**

Let \( A \) and \( B \) be densely defined linear operators in a Hilbert space \( \mathcal{H} \). Suppose that

\[
\begin{align*}
(i) & \quad D(A) \subset D(B) \\
(ii) & \quad \text{for some } a \text{ and } b \text{ in } \mathbb{R}_+ \text{ and all } \psi \in D(A),
\end{align*}
\]

\[
(a_{16.2}) \quad \| B \psi \| \leq a \| \psi \| + b \| \psi \|
\]

Then \( B \) is \( A \)-bounded. The minimum of such a
is called the relative bound of $B$ with respect to $A$. If the relative bound is zero, we say that $B$ is infinitesimally small with respect to $A$ and write $B \ll A$.

We remark that usually $b$ must be chosen large as $a$ is chosen smaller.

Sometimes it is convenient to replace (ii) with

(iii) For some $\tilde{a}, \tilde{b} \in \mathbb{R}^+$ and all $\varphi \in D(A)$

$$
(197.1) \quad \| B \varphi \|_1^2 \leq \tilde{a}^1 \| A \varphi \|_1^2 + \tilde{b}^2 \| \varphi \|_1^2
$$

Clearly if (197.1) holds, then (196.2) holds with $a = \tilde{a}$, $b = \tilde{b}$, and if (196.1) holds, then (197.1) holds with $\tilde{a}^2 = (1+\varepsilon_1) a^2$ and $\tilde{b}^2 = (1+\varepsilon^{-1}) b^2$.

Thus the infimum over all $a$ in (196.1) is equal to the infimum over all $a$ in (197.1). Note that to prove (196.2) or (197.1) it is enough to prove them on
Then (1.8.1) (Kato–Rellich)

Suppose $A$ is s.âadj, $B$ is symmetric and $B$ is A-bounded with relative bound $a < 1$, then $A + B$ is s.âadj. on $D(A)$ and a.s.a. on any core of $A$.

Further, if $A$ is bôded below by $M$, then $A + B$ is bôded below by $M - \max (\frac{1}{1-a}, a|M| + b)$,

where $a < b$ are given by (1.6.2).

Proof: We will show that if $A$ is s.âadj then

$$\text{Ran} (A + B \pm \imath \mu) = \emptyset$$

for some $\mu > 0$.

For $y \in D(A)$

we have for $\mu > 0$,

(1.8.1) \[ \|(A + \imath \mu) y\| \leq \|A y\| + \mu \| y\| \]

By the spectral theorem, $\|A + \imath \mu\| \leq 1$ and

$$\|A(A + \imath \mu)^{-1}\| \leq 1$$

Alternatively, for $\varphi = (A + \imath \mu)^{-1} \varphi$ ($1.8.1$)
But the spectral norm, for any $\mu > 0$

\[ \| A(A + i\mu) \|_\infty \leq 1 \quad \text{and} \quad \| A(A + i\mu) \|_\infty \leq \frac{1}{\mu} \]

Thus from (196.2), for $\mu > 0$ and any $4 \in \mathbb{A}$,

\[ \| B(A + i\mu) \|_\infty \leq a \| A^* (A + i\mu) A \| + b \| A^* (A + i\mu) A + I \| \leq (a + \frac{b}{\mu}) \| 4 \| \]

Thus for large $\mu = \mu_0 + C = B(A + i\mu)$, the norm is $< 1$, as $a < 1$.

Thus we set $\epsilon = \mathbf{X}$, hence $-1 + \mathbf{X} (C)$ as

so \( \text{Ran} (I + C) = \Phi \). Since $A$ is s. adj., \( \text{Ran} (A + i\mu_0) = \Phi \), also. Thus the equation

\[ (199.1) \quad (A + B + i\mu_0) \mathbf{u} = (1 + C)(A + i\mu_0) \mathbf{u}, \quad \mathbf{u} \in \text{D}(A) \]

so \( \text{Ran} (A + B + i\mu_0) = \Phi \). The case \( \text{Ran} (A + B - i\mu_0) = \Phi \) is similar. Thus by the fundamental criterion, $A + B$

is s. adj. on $\text{D}(A)$.

Now observe that $A + B$ is closable on any set.
Let $D_0$ be a core for $A$. Then it follows that $\text{Ran} \left( (A+i\mu)D_0 \right)$ is dense in $\mathbb{C}$. Hence $\text{Ran} \left( (1+C)(A+i\mu) \right)D_0$ is dense in $\mathbb{C}$ is onto. Thus $\text{Ran} \left( (A+B+i\mu)D_0 \right)$ is dense in $\mathbb{C}$ by (1.9)1. By the fundamental criteria $A+B$ is $\sigma$-a.

Finally, we prove the semi-boundedness of $A+B$. Suppose $t > 0$ and $-t < t$. Then $\text{Ran} \left( (A+t) \right) = \mathbb{C}$ and $\| B(A+t)^{-1} \| \leq \| A(A+t)^{-1} \| + b \| (A+t)^{-1} \|

Now as $A+t \rightarrow \infty + t > 0$, it follows that if \( \lambda \in \sigma(A) \), \( \frac{\lambda}{A+t} \) is a monotone function of $t$ and no \( \sup_{\lambda \in \sigma(A)} \left| \frac{\lambda}{A+t} \right| \leq \max \left( \frac{1}{\lambda A+t} , 1 \right) \)

Thus $\| B(A+t)^{-1} \| \leq \max \left( \frac{1}{\lambda A+t} , 1 \right) + b \frac{1}{A+t}$.
Suppose \( \frac{|k|}{k+n} \geq 1 \), then

\[
a \max \left( \frac{|k_n|}{k+n} \right) + b = a \frac{|k|}{k+n} + b
\]

\[
= a \frac{|k| + b}{k+n}.
\]

If \( \frac{|k|}{k+n} < 1 \), then

\[
a \max \left( \frac{|k|}{k+n} \right), \quad \frac{|k| + b}{k+n} = a + b
\]

\[
= a \frac{k+n}{k+n} + b.
\]

Thus \( ||B(A+t)|| < 1 \) if

\[
a |k| + b < k+n
\]

and if \( a(k+n) + b < k+n \), \( u < (1-a)(k+n) \geq b \).

\[
u = \frac{b}{1-a} < k+n.
\]

Thus \( ||B(A+t)|| < 1 \) if \( \max \left( \frac{b}{1-a}, \ a|k| + b \right) < k+n \)

\[
u - t < k\lambda - \max \left( \frac{b}{1-a}, \ a|k| + b \right) = M^*.
\]

For such \( t \), \( (A + B + t) = (I + B(A+t)^{-1})A + t \) is a clearly a

bijection as \( \forall \ t \in \rho(A+B) \).
We now prove that the Hamiltonian for
the hydrogen atom in $\mathbb{R}^3$

\[(203.1)\]

\[H = -\Delta - \frac{e^2}{|x|}\]

is self-adjoint on $\text{Dom}(-\Delta) = H^1(\mathbb{R}^3)$

\[= \{ f \in L^2_+: \text{ Fourier transform of } f \text{ is } L^2(\mathbb{R}^3) \}\]

Theorem (Kato)

Let $V \in L^1(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, then $-\Delta + V(x)$

is self-adjoint on $\text{Dom}(-\Delta)$.

Proof: $V \in L^1(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ means $\mathcal{F}V \in L^1$

and $V \in L^\infty$ so $V = V_1 + V_2$. For any $u \in L^\infty(\mathbb{R}^3)$

we have for $V_1, V_2$ as above,

\[(203.2)\]

\[\|V u\|_2 \leq \|V_1 u\|_2 + \|V_2 u\|_\infty \leq \|V_1\|_\frac{1}{2} \|V_2\|_\infty \|u\|_2\]

So $L^\infty(\mathbb{R}^3) \subset \text{Dom}(V)$, i.e. $\forall f \in L^1: V_2 f \in L^\infty$. Now

for any $u \in L^\infty(\mathbb{R}^3)$, and any $\varepsilon > 0$, 


\[ |\psi(x)|^2 = \frac{1}{(2\pi)^{3/2}} \int \frac{e^{ix \cdot k} \hat{\psi}(k) \, dk}{(1 + x^2)^{1/2}} \]

\[ \leq \frac{1}{(2\pi)^{3/2}} \left( \int |\hat{\psi}(k)| \, dk \right)^2 \]

\[ = \frac{1}{(2\pi)^{3/2}} \left( \int \frac{1}{(1 + x^2)^{1/2}} \, dk \right) \left( \int (1 + x^2)^{1/2} |\hat{\psi}(k)| \, dk \right)^2 \]

\[ \leq \frac{1}{(2\pi)^{3/2}} \left( \int \frac{1}{(1 + x^2)^{1/2}} \, dk \right) \left( \int (1 + x^2)^{1/2} \, \, dk \right)^2 \]

\[ \leq \frac{1}{(2\pi)^{3/2}} \left( \int \frac{1}{1 + x^2} \, \, dx \right) \left( \int (1 + x^2)^{1/2} \, \, dx \right)^2 \]

\[ = \frac{1}{(2\pi)^{3/2}} \left( \int \frac{1}{1 + x^2} \, \, dx \right)^2 \left( \int (1 + x^2)^{1/2} \, \, dx \right)^2 \]

\[ \leq \frac{1}{(2\pi)^{3/2}} \left[ \varepsilon^{-2} \left( \int (1 + x^2)^{1/2} \, \, dx \right)^2 + \varepsilon^2 \left( \int (1 + x^2)^{1/2} \, \, dx \right)^2 \right] \]

As \[ \int \frac{1}{1 + x^2} \, \, dx = c \int \frac{1}{(1 + t^2)^{1/2}} \, dt < \infty \]

we see that for any \( \varepsilon > 0 \), no matter how small, \( T > 0 \)

s.t.

\[ |\psi(x)| \leq |A| \| \psi \|_2 + \varepsilon \| \psi \|_2 \]

for \( \psi \in L^\infty(\mathbb{R}^3) \). Inserting this inequality into (203.2) we find for \( \psi \in L^\infty(\mathbb{R}^3) \)

\[ \| \psi \|_2 \leq |A| \| \psi \|_2 + \varepsilon \| \psi \|_2 + (b \| V_1 \|_2 + \| V_2 \|_\infty) \| \psi \|_2 \]
Thus \( V \) is \(-\alpha\)-bounded with arbitrarily small bound on \( L^\infty(\mathbb{R}^3) \). Since \(-\alpha\) is essentially s.a.e.i. on \( L^\infty \),
\[-\Delta + V \text{ is e.s.a. on } L^\infty(\mathbb{R}^3) \] by the Kato-Rellich Thm
and also \(-\Delta + V \text{ is s.a.e.i. on } D(-\Delta) = H^2(\mathbb{R}^3) \). \( \square \)

**Corollary 205.1**

The hydrogen Hamiltonian \( H = -\Delta - e^2/|x| \) is
e.s.a. on \( L^\infty(\mathbb{R}^3) \) and s.a.e.i. on \( D(-\Delta) = H^2(\mathbb{R}^3) \)

**Proof**

\[
\frac{1}{|x|} = \frac{1}{|x|} \sum \frac{1}{|x_i|} \frac{1}{|x_j|} \]

\[= U_v + V_v \]

and

\[
\int \frac{1}{|x|} \, d^3x < \infty, \quad U_v \in L^2. \quad \text{QED.} \]

**Theorem 205.2 (Kato's Theorem)**

Let \( \{V_k\}_{k=1}^\infty \) be a collection of real-valued functions each of which is in \( L^1(\mathbb{R}^3) + L^\infty(\mathbb{R}^3) \). Let \( V_k(y) \)
Let the mult. oper. in $L^2(\mathbb{R}^{3n})$ obtained by choosing $y_k$ to be "3 orthogonal" coordinates in $\mathbb{R}^{3n}$. Then

$$-\Delta + \sum_{k=1}^{n} V_k(y_k) \text{ is e.s.a. on } L^2(\mathbb{R}^{3n}),$$

where $\Delta$ denotes the Laplacian on $\mathbb{R}^{3n}$.

Proof by choosing $y_k = (y_{k1}, y_{k2}, y_{k3})$ to be "3 orthogonal coordinates in $\mathbb{R}^{3n}$" we mean that if $x_1, \ldots, x_{3n}$ are the standard coords in $\mathbb{R}^{3n}$, then

$$y_{kj} = \sum_{i=1}^{n} A_{ji}^{(k)} \cdot x_i$$

for some $3 \times 3n$ matrix $(A_{ji}^{(k)})$ so that the 3 vectors $v_j = (A_{j1}^{(k)}, \ldots, A_{jn}^{(k)}) \in \mathbb{R}^{3n}$ are orthonormal in $\mathbb{R}^{3n}$, $(A_{ij}^{(k)} A_{jq}^{(k)}) = \delta_{ij}, \quad 1 \leq i, q \leq 3$. Then $A$

an orthogonal matrix $O$ so $A_i^{(k)} O = e_i, \quad i = 1, \ldots, 3$

where $e_1 = (1, 0, \ldots, 0), \quad e_2 = (0, 1, 0, \ldots, 0), \quad e_3 = (0, 0, 1, 0, \ldots, 0)$

Then if $u_k = \sum_i O_i^{T} x_i$, we clearly have $y_{ki} = u_i, \quad i = 1, 2, 3$. [206]
As $O$ is orthogonal, $A$ is invariant if

$$
\sum_{\ell=1}^{3n} \frac{q_{\ell}}{\theta \, x_{\ell}} = \sum_{\ell=1}^{3n} \frac{q_{\ell}}{\theta a_{\ell}}.
$$

and clearly $L^2(\mathbb{R}^{3n})$ and $L^\infty(\mathbb{R}^{3n})$ are also invariant.

Hence for all $\varphi \in L^2(\mathbb{R}^{3n})$

$$
\int |V_k(y_k) \varphi(x)|^2 \, d^3x.
$$

$$
= \int V_k(u_1 u_2 u_3) \widehat{\varphi}(u) |^2 \, d^3u,
$$

$$
\left\| \varphi \right\|_2^2
$$

$$
\leq a^4 \int \left| -\Delta \widehat{\varphi}(u_1, \ldots, u_{3n}) \right|^2 \, du_1 \ldots du_{3n}
$$

$$
+ b^2 \int \left( \widehat{u}(u_1, \ldots, u_{3n}) \right)^2 \, du_1 \ldots du_{3n}.
$$

$$
= a^4 \int \left| (P_1^2 + P_2^2 + P_3^2) \widehat{\varphi}(P_1, \ldots, P_{3n}) \right|^2 \, dp_1 \ldots dp_{3n}
$$

$$
+ b^2 \| \widehat{\varphi} \|^2
$$

$$
\leq a^4 \int \left( \sum_{\ell=1}^{3n} P_\ell^2 \widehat{\varphi}(P_1, \ldots, P_{3n}) \right)^2 \, dp_1 \ldots dp_{3n}
$$

$$
+ b^2 \| \widehat{\varphi} \|^2
$$

$$
= a^4 \| -\Delta \widehat{\varphi} \|^2 + b^2 \| \widehat{\varphi} \|^2
$$

$$
= a^4 \| -\Delta \varphi \|^2 + b^2 \| \varphi \|^2.
$$
Thus
\[ \left( \sum_{k=1}^{m} V_k(y_k) \right) \psi \leq \left( \sum_{k=1}^{m} V_k \right) \psi \leq m \sum_{k=1}^{m} V_k \psi \leq a^2 \| \Delta \psi \|^2 + b^2 \| \psi \|^2 \]

If \( \psi \in L^\infty(\mathbb{R}^3) \), since \( a \) may be chosen as small as we like, we conclude that \( \sum_{k=1}^{m} V_k(y_k) \) is infinitesimally small and \(-A\), and the result follows by Kato-Rellich Theorem. \( \square \)

**Corollary (208.1)** (Atomic Hamiltonian)

Let \( x_1, \ldots, x_n \) in \( \mathbb{R}^3 \) be orthogonal coordinates for \( \mathbb{R}^{3n} \). Then

\[ H = -\sum_{i=1}^{n} \Delta_i - \sum_{i=1}^{n} \frac{m e^4}{|x_i|} + \sum_{i=1}^{n} \frac{e^2}{|x_i - x_j|} \]

in \( L^\infty \)

**Proof** Note, for example, that for \( i \neq j \)

\[ \sqrt{|x_i - x_j|} = \frac{1}{2} \left( \frac{|x_i|}{v_i} + \frac{|x_j|}{v_j} - \frac{|x_i - x_j|}{|x_i|} \right) \]
and
\[
\frac{x_{i1} - x_{j1}}{\sqrt{\lambda}} = \left( -1, \frac{1}{\sqrt{\lambda}}, 0, 0, \ldots, -1, 0, \frac{1}{\sqrt{\lambda}}, 0, \ldots \right) (x_{i1}, \ldots, x_{in})
\]
\[
\frac{x_{i2} - x_{j2}}{\sqrt{\lambda}} = \left( -1, 0, \frac{1}{\sqrt{\lambda}}, 0, \ldots, 0, \frac{1}{\sqrt{\lambda}}, 0, \ldots \right) (x_{i2}, \ldots, x_{in})
\]
\[
\frac{x_{i3} - x_{j3}}{\sqrt{\lambda}} = \left( \ldots, 0, 0, \frac{1}{\sqrt{\lambda}}, 0, \ldots, 0, 0, -1, \frac{1}{\sqrt{\lambda}}, \ldots \right) (x_{i3}, \ldots, x_{in})
\]
are 3 orthogonal co-ordinates in \( \mathbb{R}^{3n} \).