Exer: Note that the eigenvalues of T are simple. Why?

Lecture 13. We now show how to use min-max to prove Weyl's famous result that for $-\Delta_D^\Omega$ with Dirichlet boundary conditions in a bounded region $\Omega \subset \mathbb{R}^m$, $N(\lambda) = \dim P_{(-\infty,\lambda)}(-\Delta_D^\Omega)$ is asymptotically equal to $C_m \lambda^{m/2}\,\mathrm{Vol}\,\Omega$ times the volume of Ω where

$$C_m = \left[m\, 2^{m-1} \pi^{m/2}\, \Gamma\!\left(\tfrac{m}{2}\right)\right]^{-1}$$

where Γ is the Euler gamma function. Noting that $C_m = \tau_m/(2\pi)^m$ where τ_m is the volume of the unit ball in $\mathbb{R}^m$, we can write Weyl's result in the form

(166.1) $$N(\lambda) \sim \tau_m \lambda^{m/2} (\mathrm{vol}\,\Omega)\,(2\pi)^{-m}$$

In fact since $\tau_m \lambda^{m/2} = \mathrm{vol}\{p \in \mathbb{R}^m : p^2 < \lambda\}$ we see that

(166.2) $$N(\lambda) \sim \mathrm{vol}\{\langle x, p\rangle \in \mathbb{R}^{2m} : x \in \Omega,\ p^2 < \lambda\}/(2\pi)^m$$

Thus $(2\pi)^m N(\lambda)$ is asymptotic the volume of the region in phase space where a classical particle of mass $m = \frac{1}{2}$, moving freely in Ω (with, for exple, elastic collisions with the wall) has energy $E_{class} < \lambda$.

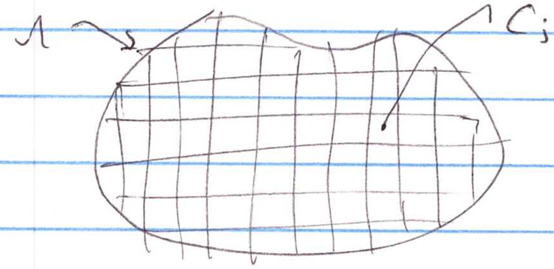
Thus we see that as $\lambda \to \infty$, the # of quantum states is given by a classical phase space volume divided by $(2\pi)^m$: this is consistent with the "old quantum theory" Bohr-Sommerfeld quantization procedure. The same is true for many other asymptotic quantum eigenvalue problems. For example, for a Schrödinger operator $H = -\Delta + W(x)$ where $W(x)$ decays suff. rapidly as $|x| \to \infty$, we have as $\lambda \to \infty$

$$\dim P_{(-\infty,0)}(-\Delta + \lambda W) \sim \frac{1}{(2\pi)^m} \operatorname{vol}\{\langle p, x\rangle : p^2 + \lambda W(x) < 0\}$$

$$= \frac{\tau_m}{(2\pi)^m} \int_{\{x : W(x) < 0\}} \lambda^{m/2} |W(x)|^{m/2} d^m x$$

See Reed-Simon Vol IV, Section XIII.15 for more discussion. We follow Section XIII.15.

The idea of the proof is to approximate Λ with cubes



On each of the cubes we first impose Dirichlet boundary conditions, $-\Delta_D^{C_i}$. Let $-\Delta_D^{\cup C_i} = \bigoplus_i (-\Delta_{Dir}^{C_i})$

Then we impose Neuman boundary conditions on the cubes, $-\Delta_N^{C_i}$ and let $-\Delta_N^{\cup C_i} = \bigoplus_i (-\Delta_N^{C_i})$

It turns out that the quadratic forms associated with $-\Delta_D^{\Lambda}$, $-\Delta_D^{\cup C_i}$ and $-\Delta_N^{\cup C_i}$ are ordered

$$(\varphi, -\Delta_N^{\cup C_i}\varphi) \leq (\varphi, -\Delta_D^{\Lambda}\varphi) \leq (\varphi, -\Delta_D^{\cup C_i}\varphi)$$

for "enough" φ's. Then by min-max

$$\mu_n(-\Delta_N^{\cup C_i}) \le \mu_n(-\Delta_D^{\Omega}) \le \mu_n(-\Delta_D^{\cup C_i})$$

Now $\mu_n(-\Delta_N^{\cup C_i})$ and $\mu_n(-\Delta_D^{\cup C_i})$ can be computed explicitly and it turns out that they agree to leading order as $n \to \infty$. It follows then that we know $\mu_n(-\Delta_\Omega^D)$ to leading order, which produces Weyl's result (166.2)

Basic message: Inserting Neuman bc's reduces the energy: inserting Dirichlet bc's increases the energy.

Question / exercise: What does this imply for wind instruments in an orchestra, say?

Definitions

Let Ω be an open region in $\mathbb{R}^n$ with connected components $\Omega_1, \Omega_2, \ldots$ (finite or infinite).

The Dirichlet Laplacian for Ω, $-\Delta_D^{\Omega}$, is the unique

s. adj. operator on $L^2(\Omega, d^m x)$ whose quadratic form is the closure of the form $q(f,g) = \int \overline{\nabla f} \, \nabla g$ with domain $C_0^\infty(\Omega)$. The Neumann Laplacian for Ω, $-\Delta_N^\Omega$, is the unique s. adj. operator on $L^2(\Omega, d^m x)$ whose quadratic form is $q(f,g) = \int \overline{\nabla f} \, \nabla g \, d^m x$ with domain

$$H^1(\Omega) = \{ f \in L^2(\Omega) : \nabla f \in L^2(\mathbb{R}^m) \}$$

where by ∇f we mean the distributional gradient, i.e.

$$\int_\Omega (\nabla f) \varphi \, dx = - \int f (\nabla \varphi) dx$$

for all $\varphi \in C_0^\infty(\Omega)$. It is easy to check that q is a closed, pos. form on $H^1(\Omega)$.

Lecture 12

If $\partial\Omega$ is "nice" it is possible to describe $-\Delta_D^\Omega$ and $-\Delta_N^\Omega$ more explicitly in ways that are useful for computation. This is true, in particular, for cubes.

Proposition 171.1 Let Ω be a cube in $\mathbb{R}^m$. Then

(a) $D_D = \{ f : f \text{ is } C^\infty \text{ up to } \partial\Omega \text{ with } f \restriction \partial\Omega = 0 \}$

is an operator core for $-\Delta_D^\Omega$ and for $f \in D_D$

$$-\Delta_D^\Omega f = -\sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}$$

(b) $D_N = \{ f : f \text{ is } C^\infty \text{ up to } \partial\Omega \text{ with } \frac{\partial f}{\partial n} \restriction \partial\Omega = 0 \}$

is an operator core for $-\Delta_N^\Omega$, and for such f,

$$-\Delta_N^\Omega f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

Proof: Wlogenerality, let $\Omega = (-1,1)^m$. (a) Let A denote the operator $-\Delta$ with domain D_D. We wish to show that $-\Delta_D^\Omega = \bar{A}$. A is clearly symmetric and by separation of variables (use multiple Fourier series) we can find a complete orthonormal

(see p 178-179 below)

basis of eigenfunctions $\{\varphi_n\}$ for A, $A\varphi_n = \lambda_n \varphi_n$ (As $n \to \infty$, $\lambda_n \to \infty$). Now if $g \in L^2(\Omega)$, set $f_N = \sum_{n=1}^{N} (\lambda_n + i)^{-1} (\varphi_n, g)\varphi_n$, $N < \infty$. Then $f_N \in D(A)$ and $(A+i) f_N = \sum_{i=1}^{N} (\varphi_n, g)\varphi_n \to g$ as $N \to \infty$, as $\{\varphi_n\}$ is an ortho. basis. Thus $\mathrm{Ran}(A+i)$ is dense (similarly $\mathrm{Ran}(A-i)$) and so A is e.s.adj. ie $\bar{A}$ is s.adj.

Let $Q(-\Delta_D)$ denote the domain of the closure of the quad. form $\int \overline{\nabla f} \nabla g$ with domain $C_0^\infty(\Omega)$. Then $-\Delta_D^\Omega$ is the (unique) s.adj operator with domain

~~is an extension of $Tf = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$, $f \in C_0^\infty(\Omega)$, with $D(-\Delta_D) \subset Q(-\Delta_D)$. As $\bar{A}f = Af = Tf$ for $f \in C_0^\infty(\Omega)$, $\bar{A}$ is a s.adj. extension of T, and so $\bar{A} = -\Delta_D$ will follow once we prove that~~

(173.0) $D(-\Delta_D^\Omega) = \{ \psi \in Q(-\Delta_D^\Omega) : \exists h \in L^2(\Omega)$ st $\int_\Omega \overline{\nabla\phi} \cdot \nabla\psi = \int_\Omega \overline{\phi}\, h \quad \forall \phi \in Q(-\Delta_D^\Omega) \}$

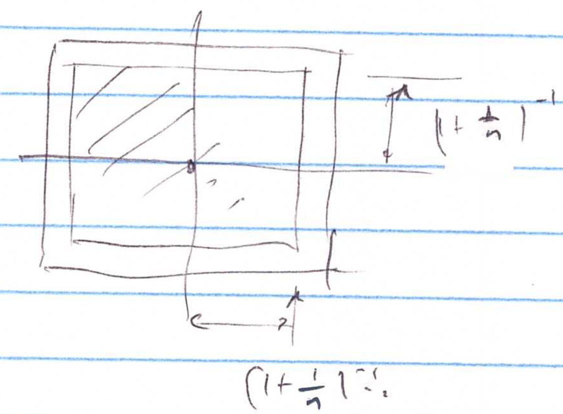
For $\psi \in D(-\Delta_D^\Omega)$, $-\Delta_D^\Omega \psi = h$. In order to conclude $-\Delta_D^\Omega = \bar{A}$ it is enough to show that $-\Delta_D^\Omega$ is an extension of A, as A is e.s.a. So suppose $\psi \in D(A)$. Then integrating by parts we obtain

(173.1) $$\int \overline{\nabla\varphi}\, \nabla\psi = \int \overline{\varphi}\, A\psi \qquad \forall \varphi \in C_0^\infty(\Omega)$$

We show that if $g \in D(A)$, $\exists\, g_n \in C_0^\infty$ st. $g_n \to g$, $\nabla g_n \to \nabla g$ in L^2, and hence $g \in Q(-\Delta_D^\Omega)$. ~~For then as $C_0^\infty(\Omega) \subset D(A)$, we have, in particular, that (173.1)~~

In particular, $\psi \in Q(-\Delta_D^\Omega)$, and as $C_0^\infty(\Omega)$ is a form core for $Q(-\Delta_D^\Omega)$, it follows that (173.1) holds $\forall \varphi \in Q(-\Delta_D^\Omega)$. Thus, by (173.0), $\psi \in D(-\Delta_D^\Omega)$ and $-\Delta_D^\Omega \psi = A\psi$.

This shows that $A \subset -\Delta_D^\Omega$ and so $\bar{A} = -\Delta_D^\Omega$. So we must show that for $g \in D(A) = \Delta_D$, such a sequence $g_n \to g$, $\nabla g_n \to \nabla g$ $\exists$. For $g \in \text{Dom } A = D_D$, let $\tilde{g}_n(x) = g((1+n^{-1})x)$ if $|x_i| \le (1+n^{-1})^{-1}$, $1 \le i \le m$, and zero otherwise



As $g(x) = 0$ for $x \in \partial\Omega$, we see that $\tilde{g}_n(x)$ is continuous and piecewise C^1 with a bded gradient. Let $g_n(x) = j_{\delta_n} * \tilde{g}_n$ for suitable small $\delta_n \to 0$, where $j(x) \in C_0^\infty(|x|<1)$, $j \geq 0$, $\int j\,dx = 1$, & $j_{\delta_n}(x) = \frac{1}{\delta_n^m} j\left(\frac{x}{\delta_n}\right)$.

j_{δ_n} is an approximation to the identity.

Then $g_n \in C_0^\infty(\Omega)$ and a standard calculation now shows that $g_n \to g$, $\nabla g_n \to \nabla g$ and so $g \in Q(-\Delta_D^\Omega)$. This shows $\bar{A} = -\Delta_D^\Omega$.

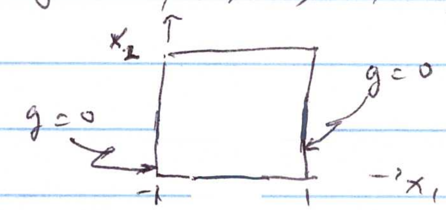
(b) Let B denote the operator $-\Delta$ with domain D_N. As in part (a) it is easy to see that B is e.s.adj by looking at its eigenfunctions. To show that $-\Delta_N^\Omega = \bar{B}$ it is again sufficient to show that $-\Delta_N^\Omega$

is an extension of B. Clearly $D(B) \subset H^1(\Omega) = Q(-\Delta_N^\Omega)$ and integrating by parts, we have for $\psi \in D(B)$

(175.1) $$\int \overline{\nabla \varphi}\, \nabla \psi = (\varphi, B\psi)$$

$\forall \varphi \in D(B)$. We show that (175.1) is true for all $\varphi \in H^1(\Omega)$. Then, by definition $\psi \in D(-\Delta_N^\Omega)$ and $-\Delta_N^\Omega \psi = B\psi$, as desired.

Let $f \in H^1(\Omega)$. We first claim that if g and Dg are continuous up to $\partial\Omega$ and $g(\pm 1, x_2, \ldots, x_n) = 0$, then



(175.2) $$(\partial_1 f, g) = -(f, \partial_1 g)$$

Suppose first that g vanishes on all of $\partial\Omega$. Then as in the proof of (a), $\exists\, g_n \in C_0^\infty(\Omega)$ st $g_n \to g$, $Dg_n \to Dg$ so that (175.2) follows for g as it holds for g_n. Now let $\{\eta_k\}$ be a sequence of C^∞

functions on Ω depending only on $x_2, \dots, x_n$ with support in $\{x : |x_j| \le 1 - \frac{1}{k}, \ j = 2, \dots, m\}$ such that $\eta_k \uparrow 1$ in Ω (as $k \to \infty$) and $\|\eta_k\|_\infty \le 1$. Then (175.2) holds for $g\eta_k$ and thus for g also as $\partial_1(g\eta_k) = \eta_k \partial_1 g$

Now for $n = (n_1, \dots, n_m)$, $n_i \ge 0$ set

(176.1) $$\Phi_n(x) = \prod_{i=1}^{m} \varphi_{n_i}(x_i)$$

where

(176.2) $$\begin{cases} \varphi_k(x) = \sin(k\pi x/2) & k = 1, 3, 5, \dots \\ \varphi_k(x) = \cos(k\pi x/2) & k = 2, 4, 6, \dots \\ \varphi_k(x) = \frac{1}{2}\sqrt{2} & , k = 0 \end{cases}$$

Then the Φ_n's are a complete orthonormal set of eigenfunctions for B in $L^2(\Omega)$. By replacing $\sin \frac{n_1 x\pi}{2}$ by $\cos \frac{n_1 x\pi}{2}$ for n_1 odd and $\cos \frac{n_1 x\pi}{2}$ by $\sin \frac{n_1 x\pi}{2}$ for n_1 even, $n_1 \ne 0$, we obtain an orthonormal family $\tilde{\Phi}_n$ (not complete!) such that

$\partial_1 \tilde{\Phi}_n = \pm \frac{\pi}{2} n_1 \Phi_n$. Note that $\tilde{\Phi}_n$ and $D\tilde{\Phi}_n$ are continuous up to the boundary and $\tilde{\Phi}_n(\pm 1, x_2, \ldots, x_m) = 0$.

Thus by (175.2) $(\partial_1 f, \tilde{\Phi}_n) = -(f, \partial_1 \tilde{\Phi}_n)$

$= \pm \frac{\pi}{2} n_1 (f, \Phi_n)$. As $\{\tilde{\Phi}_n\}$ is an orthonormal set we see that $\sum_{n_i \geq 0} n_1^2 |(f, \Phi_n)|^2 = (\frac{2}{\pi})^2 \sum_{n_i \geq 0} |(\partial_1 f, \tilde{\Phi}_n)|^2$

$\leq (\frac{2}{\pi})^2 \|\partial_1 f\|^2$, by Bessel's inequality. The same is true for $n_k (f, \Phi_n)$, $k = 2, \ldots, m$, and so

(177.0) $$\sum_{n_i \geq 0} n^2 |(f, \Phi_n)|^2 < \infty .$$

For $N \geq 0$, and for $f \in H^1$, set

(177.1) $$f_N(x) = \sum_{n^2 \leq N^2} (\Phi_n, f) \Phi_n .$$

Then $f_N \in D(B)$ and as $N \to \infty$ $f_N \to f$

On the other hand, for $N' > N$

$$\int \overline{D(f_{N'} - f_N)} \nabla (f_{N'} - f_N) = (f_{N'} - f_N, B(f_{N'} - f_N))$$

$$= \Big(\sum_{N < n^2 \le N'} (\Phi_n, f)\,\Phi_n \,,\, \sum_{N < n^2 \le N'} n^2 (\Phi_n, f)\,\Phi_n \left(\tfrac{\pi}{2}\right)^2 \Big)$$

$$= \left(\tfrac{\pi}{2}\right)^2 \sum_{N < n^2 \le N'} |(\Phi_n, f)|^2\, n^2 \quad \to 0 \quad \text{as } N' > N \to \infty$$

by (177.0). Thus $\{\nabla f_N\}$ is Cauchy and so $\nabla f_N \to h$ in $(L^2(\Omega))^m$ for some $h \in (L^2(\Omega))^m$ and as

$$(\nabla f_N, \phi) = -(f_N, \nabla\phi)$$

$\forall \phi \in C_0^\infty$, we have $(h, \phi) = (f, \nabla\phi)$

$\forall \phi \in C_0^\infty$. Thus $h = \nabla f$. This shows that

$\forall f \in H^1(\Omega)$, then $\exists f_N \in D(B)$ st $f_N \to f$ and $\nabla f_N \to \nabla f$

Taking $\varphi = f_N$ in (175.1) and letting $N \to \infty$,

we conclude that (175.1) holds $\forall f = \varphi \in H^1(\Omega)$

as desired. Thus $-\Delta_N^\Omega = \overline{B}$.

Lecture 14 On a cube $(-a, a)^m$ the eigenvectors and eigenvalues

of $-\Delta_D$ are labelled by m-tuples $n = (n_1, \dots, n_m)$

with $n_i \ge 1$, $i = 1, \dots, m$ and are given by

(178.1) $$\Phi_{n;a}(x) = a^{-m/2} \prod_{i=1}^{m} \varphi_{n_i}(x_i/a)$$