Uniqueness of $A$ follows from the following:

We now describe $D(A)$ in a more concrete fashion.

We have:

Claim: $D(A) = \{ q \circ \alpha_{A}(q) : q(4, e) = (4, x_{q}) \}$

for some $x_{q}$ and $4 \in \alpha_{A}(q)$.

Furthermore, if $4 \in D(A)$, then $A_{4} = x_{q}$.

Proof: From (13.1) we have for $4, 4' \in D(B) = D(A)$

But $D(A)$ is $\mathbb{N} \cdot 1_{+}$-dense in $\mathbb{H}_{+}$, hence (13.3) holds if

$4 \in \alpha_{A}(q)$ with $x_{q} = A_{4}$.

Thus $D(A) \subset \text{RHS of the Claim}$.

Conversely suppose for some $x_{q}$

$q(4, e) = (4, x_{q})$ \quad $4 \in \alpha_{A}(q)$

i.e.,

$(4, e)_{+1} = (4, x_{q}) + (4, e) = (4, x_{q} + e)$

But $(4, e)_{+1} = [B^{0}]e_{1}(4) \Rightarrow [B^{0}]e_{1}(4) = (4, x_{q} + e) = [j(x_{q} + e)](4)$

$4 \in \mathbb{N}(q)$. Hence $4 = B_{\rightarrow 0}j(x_{q} + e) \in D(B) = D(A)$, hence RHS $\subset D(A)$.

Also $x_{q} = j^{-1}B_{\rightarrow 0}j(x_{q} + e) = x_{q} + e$ \quad i.e. $A_{4} = x_{q}$.

Thus:

$(4, e)_{+1} = (4, x_{q} + e)$  \quad $(\ast)$

$d(4, e) = (4, A_{4}(e)) + 4 \in \mathbb{H}_{+})$.

This proves the claim.

Remark: Clearly it is enough to know $q(4, e) = (4, x_{q})$ \forall 4 in a core for $q$, to conclude that $q(4, e) = (4, x_{q})$ \forall 4 $\in \mathbb{N}(q)$.
A is uniquely determined by (133.2) in the following sense. Let \( \hat{A} \) be a self-adjoint operator with \( D(\hat{A}) \subseteq Q(\mathfrak{a}) \) and for every \( \psi \in D(\hat{A}) \)

\[
(34.1) \quad \langle \psi, \hat{A}\psi \rangle = \langle \psi, \hat{A}\psi \rangle
\]

\( \psi \) belonging to a core for \( A \). Taking limits in (34.1), we have

\[
(\psi, \hat{A}\psi) = (\psi, \hat{A}\psi)
\]

\( \psi \in Q(\mathfrak{a}) \). But then by (133.2), \( \psi \in D(\hat{A}) \) and

\[
\hat{A}\psi = \hat{A}\psi
\]

This \( \hat{A}C\hat{A} \) and so \( \hat{A} = A \) as \( A \) and \( \hat{A} \) are self-adjoint.

This completes the proof of Theorem 129.1.
Note the following: A symmetric operator always has a closed symmetric extension; but it may be that none of its closed extensions is self-adjoint. On the other hand, semi-bounded forms need not have any closed extensions, but when such extensions exist, they are the quadratic forms associated with self-adjoint operators.

Caveats:

1. As we have seen on many occasions, if $A$ and $B$ are self-adjoint, and $D(A) = D(B)$ with $B + D(A) = A$, then $A = B$. But it can happen that $a$ and $b$ are closed semi-bounded quadratic forms and $b = g(a) = a$, but without having $a = b$.

2. Let $A$ be a symmetric operator, that is, semi-bounded. Let $q$ be a quadratic form $q(x, y) = (x, A y)$ with $D(q) = D(A)$. Suppose $q$ has a closure (it always will: see later!) $\tilde{q}$, that is, the smallest closed form that extends it. Then $\tilde{q}$ is self-adjoint. A comes to $\tilde{q}$ may be bigger than $A$. The operator closure of $A$.

As noted above,

3. Quadratic forms that come directly from semi-bounded operators always have closures and hence semi-bounded operators always have self-adjoint extensions, a fact that we already knew from Corollary 100.1, where we saw that $n = n$.

The following exploits illustrate (1, 2) above.
Recall the operators $T_0, T_{a,b}, T_{a,b,∞}$ defined earlier as $-\frac{d^2}{dx^2}$ acting on

\[
T_0 = \{ f \in AC^2[0,1] : f(0) = f(1) = 0 = f'(0) = f'(1) \}
\]

\[
T_{a,b} = \{ f \in AC^2[0,1] : a f(0) + f'(0) = 0 \}
\]

\[
T_{a,b,∞} = \{ f \in AC^2[0,1] : f''(0) = f''(1) = 0 \}
\]

respect. $T_0$ is closed and symmetric, but not self-adjoint. $T_{a,b}$ and $T_{a,b,∞}$ are self-adjoint.

(i) Then if $q_0(x, y) = (4, T_{0,4})$ for $0, 4 \in D_0$, the form $q_0$ has a smallest closed extension $\tilde{q}_0 = q_{0,\tilde{0}}$.

\[
Q(q_{0,\tilde{0}}) = \{ f \in AC[0,1] : f \leq 0, f(0) = f(1) = 0 \}
\]

This extension is the form associated with $T_{0,∞}$, which illustrates

No remark (2) above.

(ii) The form $q_{a,b}$ associated with $T_{a,b}$ (see details below) strictly

has a form domain $Q(q_{a,b})$ which contains the form

\[
Q(q_{0,∞}) \cup Q(q_{∞,0}) \cup Q(q_{0,0}) = 0
\]

This illustrates remarks (i) above.
Let 
\[ Q(a,b) = \{ \varphi \in AC([0,1]) : \varphi \in L^1([0,1]) \} \]

(37.0) 
\[ d_{a,b}(\varphi,\psi) = \int_0^1 \varphi \psi' \, dx = a \varphi(0) + b \varphi(1) \]

for \( a, b \in Q(a,b) \)

To see that \( d_{a,b} \) is closed on \( Q(a,b) \) note that

for \( \varphi \in Q(a,b) \)

\[ (4) \quad \varphi(x) = \varphi(0) + 2 \int_0^x \varphi'(t) \, dt \quad \text{at} \quad 0 \leq x, y \leq 1 \]

and no for any \( \varepsilon > 0 \)

\[ |\varphi(x)|^2 < |\varphi(y)|^2 + \varepsilon \int_0^1 |\varphi'(t)|^2 \, dt \]

and then integrating with \( y \) we find

(37.1) 
\[ |\varphi(x)|^2 \leq (1 + \varepsilon) \int_0^1 |\varphi'|^2 \, dt + \varepsilon \int_0^1 |\varphi'|^2 \, dt \]

Setting \( x = 0 \) or \( y \) in (37.1), it is easy to see

(exercise) that \( d_{a,b} \) is closed on \( Q(a,b) \), and semi-"barded.

Now let \( \psi \in D(A) \), where \( A \) is the same operator associated with \( d_{a,b} \) by Thm (2a.).
Then \( u + e \in \Omega(\alpha,n) \)

\[(138.1) \quad d(u, u^*) = (u, h) \]

For some \( h \in L^2 \) and \( h = A u \)

In particular, \((138.1)\) is true \( \forall u \in L^2(0,1) \)

For such \( u \), \( d(u, u^*) = \int_0^1 u' + u \) and

\[(4,4) = \int_0^1 u' = -\int_0^1 h \text{ where } h(x) = \frac{d}{dt} h(t) \text{ at } t = x \]

and so \( \int_0^1 u' (u' + u) = 0 \) \( \forall u \in L^2(0,1) \).

As before, \( \text{Thus } u + t h = \text{ constant } \), hence \( u \).

\( u \in A - C_2(0,1) \) and \( u'' = -H' = -h \in L^2 \)

\( u \in A - C_2(0,1) \) and \( u'' = -H' = -h \in L^2 \)

Thus, \( D(A) \subset A \) \( 4 \in \{ 4'' \} \), \( 4'' = -H' = -h \in L^2 \)

Then \( D(A) \subset A \) \( 4 \in \{ C^2(0,1) \} \), \( \forall 4'' = -H' = -h \in L^2 \)

Now, for \( 4 \in D(A) \), we have for any \( \phi \in \Omega(\alpha,n) \)

\[d(u, u^*) = \int_0^1 u' + u - \phi(t) \bigl( 0, u_0 + b \phi(1) \bigr) \]

\[= \int_0^1 u' - \phi(0) \int_0^1 u' \]

Integrating \( \text{LHS by parts we find} \)
\[ - \frac{1}{\bar{u}} \varphi'' = \frac{1}{\bar{v}} \varphi' - \varphi(1)\varphi'(1) + \varphi(0) \cdot \varphi'(0) \]

Now for \( q \in Q(4, a, b) \) let \( \varphi(1) = 0 \), \( \varphi(0) \neq 0 \).

We see that we must have

\[ \varphi(0) \varphi'(0) = -a \varphi(0) \varphi(0) \]

and no

\[ \varphi'(0) + a \varphi(0) = 0 \]

Similarly from \( \varphi(0) \neq 0 \), \( \varphi(1) \neq 0 \), we obtain

\[ \varphi'(1) + b \varphi(1) = 0. \]

Thus

\[ D(A) \subset \{ Ac^2(0, 1) : \varphi(1) + b \varphi'(1) = 0 \} \]

On the other hand, it is easy to see that if

\[ q \in D(A) \cap \{ \text{real} \} \]

Then

\[ q(4, i) = (4, -4) \quad 4 \neq Q(4, a, b) \]

and no \( q \in D(A) \). Thus

\[ Aq = -4q \quad \text{with domain} \]

\[ D(A) = \{ Ac^2(0, 1) : \varphi'(1) + b \varphi(1) = \varphi'(0) + a \varphi(0) = 0 \} \]
As already noted, it follows from Corollary 100.1, that a semi-bounded symmetric operator \( A \), \((\varphi, A\varphi) > -\infty \) for equal deficiency indices and hence such an operator always has s. adj. extensions. It turns out that there is a distinguished extension, called the Friedrichs extension, which is obtained from the quasi-form associated with \( A \).

**Theorem 140.1 (The Friedrichs Extension)**

Let \( A \) be a positive symmetric operator and let

\[
(4.0.1) \quad \phi(\varphi, \varphi) = (\varphi, A\varphi)
\]

for \( \varphi, \psi \in D(A) \). Then \( \phi \) is a closable form and its closure \( \hat{\phi} \) is the quasi-form of a unique s. adi operator \( \hat{A} \), \( \hat{A} \) is a pos. extension of \( A \), and the lower bound of its spectrum is the lower bound of \( \phi \). Further, \( \hat{A} \) is the only s. adi extension of \( A \) whose domain is contained in the form domain of \( \phi \).
Notation: \( q(uv) = q(u, v) \), \( u, v \in \mathbb{Q}(q) \).

**Definition 14.1.0** A semi-bounded form \( q \) is **closable** if

\[
\lim_{n \to \infty} q(u_n - u_m) = 0
\]

where \( u_n, u_m \to 0 \).

**Lemma 14.1.2**

A semi-bounded form \( q \) is closable if and only if \( q \) has a closure (the smallest closed extension) \( \hat{q} \) defined in the following way. \( \hat{q}(\hat{q}) \) is the set of all \( u \) for which the set \( \{ q(u) \} \) is closed. Then

\[
\hat{q}(u, v) = \lim_{n \to \infty} q(u_n, v_n)
\]

for any \( u_n \to u, q(u_n - u_m) \to 0 \) and \( v_n \to v, q(v_n - v_m) \to 0 \).

**Remark:** We have seen (see p.128) that a semi-bounded quadratic form may not have a closed extension.

**Proof:** Let \( q \) be a closed ext. of \( q \). Then

if \( u_n \to 0 \), \( q(u_n - u_m) \to 0 \), and \( u_m \in \mathbb{Q}(q) \), then

\[
\lim_{n \to \infty} q(u_n - u_m) = 0
\]

and no by the closedness of \( q \),

\[
q(u_n) = q(u) = q(u - 0) = 0
\]

This proves necessity. (Why?)

Wlog assume \( q \geq 0 \). Hence we have C. Schwert,
\[ d(u, v) \leq q \frac{1}{2}(u) \cdot q \frac{1}{2}(v) \] and the triangle inequality, \[ d(\bar{u} + \bar{v}) \leq q \frac{1}{2}(\bar{u}) + q \frac{1}{2}(\bar{v}) \]

Conversely, to prove sufficiency, let \( q(\bar{\eta}) \) be defined as in the lemma. Then for \( u_n, v_n \) as above

\[ |d(u_n, v_n) - d(u_n, v_{n-1})| \]

\[ \leq |d(u_n - u_{n-1}, v_n)| + |d(u_{n-1}, v_n - v_{n-1})| \]

\[ \leq |d(u_n - u_{n-1}, u_{n-1})|^{\frac{1}{2}} |d(u_{n-1}, v_{n-1})|^{\frac{1}{2}} + |d(u_{n-1}, u_{n-1})|^{\frac{1}{2}} |d(v_n - v_{n-1}, v_{n-1})|^{\frac{1}{2}} \]

Note \( d(u_n - u_{n-1}, u_{n-1}) \to 0 \) and thus

\[ d(u_n, v_n) \leq \left( d(u_{n-1}, v_{n-1}) \right)^{\frac{1}{2}} \left( d(u_{n-1}, u_{n-1}) \right)^{\frac{1}{2}}. \]

we see that \( d(u_n, v_n) \) is bounded. Similarly for

\[ d(u_{n-1}, v_{n-1}) \to d(u, v) \] as \( n \to \infty \). It follows

that \( \{ \sqrt{d(u_n, v_n)} \} \) is Cauchy and hence

\[ \left( \lim_{n \to \infty} d(u_n, v_n) \right)^2 \]
The limit in (142.1) is independent of the seq's
\(u_n\)'s, \(v_n\)'s. Indeed, let \(u_n'\)s, \(v_n'\)s be other sequences s.t.
\(u_n'\to u_n, \ q(u_n'-u_n, v_n'-v_n)\to 0\) as \(n\to\infty\), \(q(u_n'-v_n', u_n'-v_n)\to 0\).

Then \(u_n'-u_n\to 0\) and

\[q'(u_n'-u_n, v_n'-v_n) = \left(\frac{q(u_n'-u_n, v_n'-v_n)}{q(u_n'-u_n, v_n'-v_n)}\right)^{1/2}\]

\[\leq q'(u_n'-u_n, u_n'-v_n) + q'(v_n'-u_n, v_n'-v_n)\to 0\] as \(n\to\infty\).

And hence by the assumptions on \(q\),

\[q'(u_n'-u_n, u_n'-u_n)\to 0\]

and similarly for \(u_n'-v_n\).

Hence \(q(u_n', v_n') = q(u_n'-u_n, v_n') + q(u_n, v_n'-v_n) + q(u_n, v_n) = q(u_n, v_n) + o(1)\)

This proves that \(\tilde{q}\) is well-defined by (142.1).

We note that
(143.1) \(\tilde{q}(u_n-u)\to 0\) if \(u_n\to u\), \(q(u_n-u_m)\to 0\).

In fact by (141.2) applied to \(u_n-u\),

\[\lim_{n\to\infty} \tilde{q}(u_n-u) = \lim_{n\to\infty} \tilde{q}(u_n-u_m) = 0\]
Clearly, by construction, \( \hat{\phi} \) is semi-totaled and \( \hat{\phi} \) is an extension of \( \phi \). We show that \( \hat{\phi} \) is closed.

Suppose \( u_n \in \phi(\hat{\phi}) \) and

\[
\|u_n - u\| \leq \frac{1}{n}, \quad \hat{\phi}^\prime(\|u_n - u\|) \to 0 \text{ as } n, m \to \infty.
\]

Then by (14.1.1), for each \( n \), \( \forall u_n \in \phi(\hat{\phi}) \)

\[
\|u_n - u\| \leq \frac{1}{n}, \quad \hat{\phi}^\prime(\|u_n - u\|) \leq \frac{1}{n}.
\]

Hence \( \|u_n - u\| \leq \frac{1}{n} + 11\|u_n - u\| \to 0 \)

\[
c_n \phi(\|u_n - u\|) = \hat{\phi}^\prime(\|u_n - u\|) \leq \hat{\phi}^\prime(\|u_n - u\|) + \hat{\phi}^\prime(\|u_n - u\|)
\]

\[
\leq \frac{1}{n} + \frac{1}{m} + \hat{\phi}^\prime(\|u_n - u\|) \to 0 \text{ as } n, m \to \infty.
\]

Hence \( \hat{\phi} \) is closed. Clearly \( \hat{\phi} \) is the smallest closed extension of \( \phi \).

Proof of Thm 140.1

We must show that \( \phi \) is closable on \( D(A) \)

\[
i.e. u_n \to \phi(\|u_n - u\|) \to 0, u_n \in D(A) \Rightarrow \phi(\|u_n\|) \to 0.
\]
\[ p(u_n) = p(u_n, u_n) \]
\[ = p(u_n, u_n - u_m) + p(u_n, u_m) \]

Thus
\[ |p(u_n)| \leq q_u(u_n) q_u^2(u_n - u_m) \]
\[ + |p(u_n, u_m)| \]
\[ = q_u^2(u_n) q_u^2(u_n - u_m) \]
\[ + |(A u_n, u_m)| \]
\[ \leq q_u^2(u_n) q_u^2(u_n - u_m) + ||A u_n|| u_m|| \]

Let \( \tau \) be given.

Now given \( \epsilon > 0 \) we can find \( N \) such that for \( n, m > N, \)
\[ q_u^2(u_n - u_m) \leq \epsilon. \]

Hence for \( m > n > N, \) as \( q_u^2(u_n) \) is bounded, say \( q_u^2(u_n) \leq C, \)
\[ |p(u_n)| \leq C \epsilon \leq ||A u_n|| u_m|| \]

Let \( m \to \infty \) we see that \( |p(u_n)| \leq C \epsilon \) if \( p(u_n) \to 0. \) Thus \( p \) is closable, and hence has a closed extension. We conclude that there exists a $\sigma$-adj.
operator \( \hat{A} \) with \( D(\hat{A}) \subseteq \mathbb{Q}(q) \). Moreover

\[ q \in D(\hat{A}) \iff \]

\[ \hat{q}^1(q, q) = (q, h) \quad \forall q \in \mathbb{Q}(q). \]

can then

\[ \hat{A} \cdot q = h. \]

We now show that \( \hat{A} \) is an extension of \( A \)

If \( q \in D(A) \), then by construction

\[ (\hat{A} \cdot q, q) = q \cdot (q, q) \]

and hence

\[ (\hat{A} \cdot q, 1) = \hat{q}^1(q, q) \quad \text{(why?)} \]

for any \( q \in \mathbb{Q}(q) \) as \( D(A) \) is a cone for \( \hat{A} \). In particular for

\[ q \in D(\hat{A}), \quad (\hat{A} \cdot q, 1) = \hat{q}^1(q, q) = (q, \hat{A} \cdot q) \]

Hence \( q \in D(\hat{A}^*) = D(A^*) \) \( \hat{A}^* \cdot q = \hat{A} \cdot q = A \cdot q \)

So \( \hat{A} \) extends \( A \), as claimed above.

The statement about the spectrum of \( \hat{A} \) is left as an

easy exercise. \( \square \)
We now show that $\hat{A}$ is the unique s.adj extension of $A$ with $D(\hat{A}) \subset Q(\hat{A})$. Indeed, suppose $\hat{A}$ is a s.adj ext. of $A$ with $D(\hat{A}) \subset Q(\hat{A})$.

For $y \in D(\hat{A})$ and $q \in D(A) \cap D(\hat{A}^\ast)$, have

$$(y, \hat{A}q) = (\hat{A}q, y) = (Aq, y)$$

Now for any $p, q \in D(A) \subset Q(\hat{A})$.

$$\hat{q}(p, q) = (Aq, q)$$

and as $D(A)$ is a core for $Q(\hat{A})$ we have

$$\hat{q}(p, q) = (Aq, q) \quad \forall p \in D(A), \quad q \in Q(\hat{A}).$$

In particular, for $p = q$ and $q = 4 \in D(\hat{A}) \subset Q(\hat{A})$, we have

$$\hat{q}(q, q) = (Aq, q)$$

Again noting that $D(A)$ is a core for $\hat{q}$, we conclude that $\hat{q}(q, q) = (Aq, q) \quad \forall q \in Q(\hat{A})$.

But then by (139.2), $q \in D(\hat{A})$ and $\hat{A}q = \hat{A}q$.

Thus $\hat{A}C \hat{A}$ and as both are s.adj ops, $\hat{A} = \hat{A}$. 

Example

Let \( A = -u''/ax \) with domain \( \mathcal{D}(A) = C^0([0,1]) \).

\( u \in H^2(0,1). \) Then

\[
q(u,v) = \int_0^1 \overline{u'} A v = \int_0^1 \overline{u'}(v')
\]

\[
= \int_0^1 \overline{u'} v', \quad u, v \in \mathcal{D}(A)
\]

\( \equiv 0. \) Hence \( q \) is closable and (exercise)

\[
\hat{q}(u) = \int_0^1 |u'|^2 \, dx, \quad Q(q) = \frac{3}{2} u + A(u)(0,1) : u' \in L^2
\]

\( u(0) = u(1) = 0 \)

Also the Friedrichs extension \( H \) of \( A \) has domain (exercise)

\[
\mathcal{D}(H) = \{ f \in L^2 : f \in AC^1[0,1], \ f'' + f' \in L^2, \ f(0) = f(1) = 0 \}
\]

\[
H \equiv -D'' \text{ for } f \in \mathcal{D}(H)
\]

This is the Dirichlet operator.

If we take

\[
\hat{q}(u,v) = \int_0^1 \overline{u'} v', \quad \text{for } u, v \in C^0[0,1]
\]

Then

\[
\hat{q}(u,v) = \int_0^1 \overline{u} (-v'') = \int_0^1 \overline{u} (A v)
\]

when \( \text{dom } A = \{ f \in C^0(0,1) : f(0) = f'(1) = 0 \}, A f = -f' \).
The Friedrichs extension of $A$ is the Neumann Operator.

**Example 1** (Weak solutions of PDE's)

Let $\Omega$ be an open region in $\mathbb{R}^n$ and let $A$ be the operator $-\Delta + I$ with domain $\mathcal{C}^0(\Omega) \subseteq \mathcal{L}^2(\Omega)$.

$A$ is symmetric and bounded below by 1. If $A^*$ is the Friedrichs extension of $A$, then $A^* \geq I$ and

$$\mathop{\text{Dom}} A^* = L^2(\Omega).$$

Thus for any $g \in L^2(\Omega)$, $f \in D(A^*)$ so that $f = g$.

$(\Omega \setminus \overline{\Omega})$ is a core for $A$.

Now as

$$f_n \in \mathcal{C}^0(\Omega), \quad f_n \to f$$

in $L^1(\Omega)$, so that for any $u \in \mathcal{C}^0(\Omega)$:

$$\frac{\partial}{\partial x^i} u = \mathcal{A} u,$$

we have

$$\lim_{n \to \infty} \int u(-\Delta + I) f_n = \int (-\Delta + I) u f_n = \int (\Delta + I) u f_n$$

But $\mathcal{A}^* u = \mathcal{A} f = (u, f)$. Thus, for any $g \in L^2$, the equation $(-\Delta + I) f = g$ has a weak
solution in $L^1(\mathbb{R})$. Since $Af = f - g \in L^1(\mathbb{R})$
we can use Schauder Theory to show a certain amount
of regularity for $f$. If $g \in C^\infty_0(\mathbb{R})$, this applying
$\Delta$ repeatedly to the eqtn $Af = f - g$ proves
$f \in \bigcap_{m=1}^\infty W^m(\mathbb{R})$, so in this case Schauder $Th^* =$
$f \in C^\infty_0$.

**Example:**

Let $A$ be a closed, densely defined operator and

$\text{let } D(A^2) = \{ u \in D(A) : A^4 u \in D(A) \}$

Then $A^2$ on $D(A^2)$ with $A^2 A u = A^4 (A u)$
is s. adj.

**Remark:** It is not evident a priori that $D(A^2)$
contains any vectors other than the zero vector.

**Proof of Example:**

$q(u, v) = (A^2 u, A^2 v)$ is a

positive quad. form on $D(A) \times D(A)$. Also as $A$ is a
closed operator, \( q \) is in a closed form.

Hence if a regular operator \( T \) with

\[
D(T) \subset Q^*(q) = D(A^*)
\]

such that \( q \in D(T) \iff q(v, u) = (q, h), \quad \forall u \in D(A^*) \) for some \( h \in \psi \). In this case, \( h = T u \).

But if \( q(v, u) = (A v, A u) = (q, h), \quad \forall u \in D(A) \)

then \( A u \in D(A^*) \) and \( A^* A u = h = T u \),

hence \( D(T) \subset D(A^* A) \) and \( T u = A^* A u \) if \( u \in D(T) \).

Conversely if \( u \in D(A^* A) \) then \( A u \in D(A^*) \)

and \( q(v, u) = (A v, A u) = (q, A^* A u) \) if \( u \in D(A) \)

Thus \( u \in \text{Dom } T \) and \( T u = A^* A u \) since \( D(A^* A) \subset D(T) \).

\[
\text{Lecture 12}
\]

\textbf{Example 3}

Let \( \psi = L^1(0,1) \) and let \( u(x) \in L^1(0,1) \), \( u \) real valued.