and $u_+(0) + \chi_u u_-(0) = 0 \quad , \quad u_+(1) + \chi_u u_-(1) = 0$

$v_+(0) + \chi_v v_-(0) = 0 \quad , \quad v_+(1) + \chi_v v_-(1) = 0$

But $u_\pm(1) = u_\pm(0)$ and $v_\pm(1) = -v_\pm(0)$ so it is enough to verify $u_+(0) + \chi_u u_-(0) = (1+e^{\lambda_+}) + \chi_u(1+e^{\lambda_-}) = 0$ and

$$v_+(0) + \chi_v v_-(0) = (1-e^{\lambda_+}) + \chi_v(1-e^{\lambda_-}) = 0$$

and so

(119.0) $\chi_u = -\dfrac{1+e^{\lambda_+}}{1+e^{\lambda_-}}$, $\chi_v = -\dfrac{1-e^{\lambda_+}}{1-e^{\lambda_-}}$. As $\lambda_- = \bar{\lambda}_+$, we clearly have

$$|\chi_u| = |\chi_v| = 1 .$$

This completes the construction of U with the desired properties. $\square$

Exercise: Find U for $T_{a,b}$, $-\infty < a, b < \infty$ and T_{per}.

$$\cancel{\|u_+\|^2 = \|a u_- + b v_-\|^2}$$

$$\cancel{\|v_+\|^2 = \|c u_- + d v_-\|^2}$$

$$\cancel{(u_+, v_+) = (a u_- + b v_-, c u_- + d v_-).}$$

Lecture 10

Another very elegant application of von Neumann's theorem concerns the Hamburger moment problem:

Let ρ be a pos. measure on $\mathbb{R}$ (with finite moments) and define

(119.1) $$a_n = \int x^n \, d\rho(x) \quad , \quad n = 0, 1, 2, \ldots$$

The #'s a_n are called the moments of the meas. ρ.

The Hamburger moment problem is to determine conditions on a sequence of real #'s $\{a_n\}_{n \geq 0}$ so that $\exists$ a meas. satisfying (119.1)

Theorem 120.1

A sequence of real #'s $\{a_n\}$ are the moments of a pos. meas. on $\mathbb{R}$ $\Leftrightarrow$ $\forall N$ and all $\beta_0, \beta_1, \ldots, \beta_N \in \mathbb{C}$

(120.2) $$\sum_{n,m=0}^{N} \bar{\beta}_n \beta_m a_{n+m} \geq 0$$

Proof: First suppose that ρ is a pos. meas and (119.1) holds. Then

$$\sum_{n,m=0}^{N} \bar{\beta}_n \beta_m a_{n+m} = \int_{-\infty}^{\infty} \left| \sum_{n=0}^{N} \beta_n x^n \right|^2 d\rho(x) \geq 0$$

Conversely, suppose that (120.2) holds. Let P be the set of poly.'s on $\mathbb{R}$ with complex coefficients and define the sesquilinear form on P by

$$\left(\sum_{n=0}^{N} \beta_n x^n , \sum_{m=0}^{M} \alpha_m x^m \right) = \sum_{m=0}^{M} \sum_{n=0}^{N} a_{n+m} \bar{\beta}_n \alpha_m$$

By (120.2) the form is non-negative. The standard argument shows that we must have the Schwartz inequality

(121.1) $$|(\pi, \sigma)| \leq (\pi, \pi)^{\frac{1}{2}} (\sigma, \sigma)^{\frac{1}{2}}$$

for all $\pi, \sigma \in P$.

Let $Q = \{ \pi \in P : (\pi, \pi) = 0 \}$

Necessarily Q is a linear subspace of P for if $\pi, \sigma \in Q$, then

$$(\alpha\pi + \beta\sigma, \alpha\pi + \beta\sigma)$$

$$= |\alpha|^2 (\pi, \pi) + |\beta|^2 (\sigma, \sigma)$$

$$+ \bar{\alpha}\beta (\pi, \sigma) + \alpha\bar{\beta} (\sigma, \pi)$$

$$= \bar{\alpha}\beta (\pi, \sigma) + \alpha\bar{\beta} (\sigma, \pi)$$

$$= 0, \quad \text{by } (121.1).$$

Let $\mathcal{H}$ be the Hilbert space obtained by completing P/Q in the inner product

(122.1) $$(\tilde{\pi}, \tilde{\sigma})_{\mathcal{H}} \equiv (\pi, \sigma)$$

for $\tilde{\pi} = \pi + Q$, $\tilde{\sigma} = \sigma + Q$. Note that, again by (121.1), $(\tilde{\pi}, \tilde{\sigma})_{\mathcal{H}}$ is a well-defined inner product on P/Q: $(\tilde{\pi}, \tilde{\pi})_{\mathcal{H}} \geq 0$ $\forall\ \tilde{\pi} \in P/Q$ and $(\tilde{\pi}, \tilde{\pi}) = 0 \Leftrightarrow \tilde{\pi} = 0$

Consider the map $A : P \to P$ defined by

$$A : \sum_{0}^{N} \beta_n x^n \to \sum_{n=0}^{N} \beta_n x^{n+1}$$

It is easy to see that A is sym. and $A : Q \to Q$, again by the Schwartz inequality

$$|(A\psi, A\psi)| = |(A^2\psi, \psi)| \leq (A^2\psi, A^2\psi)^{\frac{1}{2}} (\psi, \psi)^{\frac{1}{2}}$$

So $\psi \in Q \Rightarrow A\psi \in Q$. Thus A drops down to a sym. oper $\tilde{A}$ on $\mathcal{H}$ with domain P/Q,

$$\tilde{A}(\pi + Q) = \hat{A}\pi + Q.$$

If C denotes the usual complex conj on P, then it is easy to see that C drops down to a conjug. $\tilde{C}$ on $\mathcal{H}$ and $\tilde{A}\tilde{C} = \tilde{C}\tilde{A}$. Thus $\tilde{A}$ has some s. adj. extension, call it $\hat{A}$. Let ρ be the spec. meas. for the vector $\tilde{1} = 1 + Q$ in $P/Q \subset \mathcal{H}$. Then

$$\int x^n \, d\rho(x) = (\tilde{1}, \hat{A}^n \tilde{1})$$

$$= (\tilde{1}, \tilde{A}^n \tilde{1})$$

$$= (1, A^n 1)$$

$$= a_n. \qquad \text{QED}.$$

In general ρ is not unique. However (see RS II p205-206, if $|a_n| \leq C D^n n!$, $\forall n$, for some $C, D > 0$, then ρ is unique.

A very important aspect of Hilbert space theory is the relationship between operators and quadratic forms. (see RS Vol I p276 et seq.)

One consequence of the Riesz representation theorem is that there is a 1-1 corresp. between bounded quadratic forms and bounded operators! That is, any sesquilinear map $q : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ which satisfies

(124.1) $$|q(\varphi, \psi)| \leq M \|\varphi\| \|\psi\|$$

is of the form $q(\varphi, \psi) = (\varphi, A\psi)$ for some bded oper. A. The situation is more complicated if one removes the bdedness restriction.

<u>Definition 124.2</u> A <u>quadratic form</u> is a map

$$q : Q(q) \times Q(q) \to \mathbb{C}$$

where $Q(q)$ is a dense linear subspace of $\mathcal{H}$ called the <u>form</u>

domain, such that $q(\cdot, \psi)$ is conjugate linear and $q(\varphi, \cdot)$ is linear for $\varphi, \psi \in Q(q)$. If $q(\varphi,\psi) = \overline{q(\psi,\varphi)}$ we say that q is symmetric. If $q(\varphi,\varphi) \geq 0 \;\forall \varphi \in Q$, q is called positive and if $q(\varphi,\varphi) \geq -M\|\varphi\|^2$ for some M, we say that q is semi-bounded. Note that by polarization, if $\mathcal{H}$ is complex, then

$$q \text{ semi-bounded} \Rightarrow q \text{ symmetric}.$$

Exple 1 Let $\mathcal{H} = L^2(\mathbb{R})$ and $Q(q) = C_0^\infty(\mathbb{R})$ with $q(f,g) \equiv \overline{f(0)}\, g(0)$. One could formally write

$$q(f,g) = (f, Ag)$$

where $A : g \to \delta(x)\, g(x)$, δ = delta funct. at 0.

Since mult. by $\delta(x)$ is not an operator, q is an example of a quad. form not likely to be assoc. with an operator.

Example 2 Let $A = A^*$ on $\mathcal{H}$: By the spectral Th^m, A is isomorphic to mult. by x on $\bigoplus_{n=1}^{N} L^2(\mathbb{R}, d\mu_n)$.

Let

$$Q(q) = \left\{ \{\psi(x,n)\}_{n=1}^{N} : \sum_{n=1}^{N} \int_{-\infty}^{\infty} |x|\, |\psi(x,n)|^2 \, d\mu_n(x) < \infty \right\}$$

and for $\psi, \phi \in Q(q)$

$$q(\varphi, \psi) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} x\, \overline{\varphi(x,n)}\, \psi(x,n)\, d\mu_n(x).$$

We call q the <u>quadratic form</u> assoc. with A and write $Q(q) = Q(A)$; $Q(A)$ is called the <u>form domain of the operator A</u>. For $\varphi, \psi \in Q(A)$ we will often write

$$q(\varphi, \psi) = (\varphi, A\psi)$$

even though A does not make sense on all $\psi \in Q(A)$.

$Q(A)$ is in some sense the largest domain on which q is defined. Note that we always have $D(A) \subset Q(A)$.

We need to extend the notion of "closed" from operators to forms. Recall that A is closed $\Leftrightarrow$ its graph is closed which is the same as saying $D(A)$ is closed under the norm $\|\psi\|_A = \|A\psi\| + \|\psi\|$. Analogously we

Definition: Let q be a semibdd quad. form, $q(\psi,\psi) \geq -M\|\psi\|^2$. We say q is closed if $Q(q)$ is complete under the norm

(127.1) $$\|\psi\|_{+1} = \sqrt{q(\psi,\psi) + (M+1)\|\psi\|^2}$$

Exercise (127.2): Show that $\|\cdot\|_{+1}$ is indeed a norm

Exercise (127.3): Show that

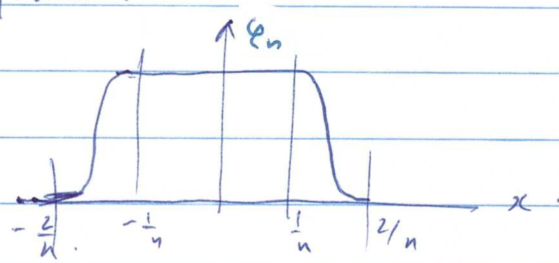
q is closed $\Leftrightarrow$ $\left[\psi_n \in Q(q),\ \psi_n \xrightarrow{\#} \psi,\ q(\psi_n - \psi_m, \psi_n - \psi_m) \to 0 \text{ as } n,m \to \infty \Rightarrow \psi \in Q(q) \text{ and } q(\psi_n - \psi, \psi_n - \psi) \to 0\right]$.

Exercise (127.4) Show that the quadratic form associated with a semi-bounded self-adjoint operator A (see exple 2 above) is closed. Moreover if $D \subset D(A)$ is a core for A ie $\overline{A \restriction D} = A$ then D is a core for q ie if $\psi \in Q(A)$, then $\exists\, \psi_n \in D$ such that $\|\psi_n - \psi\|_{+1} \to 0$. Exercise: $D(A)$ is a core for q.

Insert from p 128+ →

Consider Exple 1, $q(f,g) = \overline{f(0)}\, g(0)$, $f, g \in Q(q) = C_0^\infty(\mathbb{R})$.

Suppose q has a closed extension $\tilde{q}$ with domain $\tilde{Q}$. Then if $\varphi_n \in \tilde{Q}$, $\varphi_n \xrightarrow{\mathcal{H}} \varphi$ and $\tilde{q}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \to 0$ then $\varphi \in \tilde{Q}$ and $\tilde{q}(\varphi_n - \varphi, \varphi_n - \varphi) \to 0$. Let $\varphi_n \in C_0^\infty(\mathbb{R}) \subset \tilde{Q}$ be defined as follows:



Then $\varphi_n \to 0$ in $\mathcal{H}$ and $\tilde{q}(\varphi_n - \varphi_m, \varphi_n - \varphi_m) = 0$. But $0 \in C_0^\infty(\mathbb{R}) \subset \tilde{Q}$ and
$\tilde{q}(\varphi_n - \varphi, \varphi_n - \varphi) = q(\varphi_n, \varphi_n) = 1 \not\to 0$. Thus q has, even though q is pos (& ∴ symmetric), no closed extension. In particular, there cannot be a semi-bded s.adj. op. A st $q(f,g) = (f, Ag)$ for $f, g \in C_0^\infty(\mathbb{R}) \subset D(A)$.

The deep fact about semi-bded quadratic forms is that, unlike the case for operators, they cannot be closed & symmetric, Insert on p.128

<u>Remark</u> In general we do not know that the quadratic form associated with a s. adj. operator A is closed if A is not semi-bounded. For suppose A is multiplication by x in $L^2(\mathbb{R}, dx)$. Then $Q(A) = \{ f \in L^2(\mathbb{R}, dx) : \int |x| |f(x)|^2 \, dx < \infty \}$. Now let $g \in L^2(\mathbb{R}, dx)$, but $g \notin Q(A)$, g even. Let $g_n(x) = \chi_{(-n,n)}(x) \, g(x)$, where $\chi_{(-n,n)}$ is the charac. func. of the set $(-n, n)$. Note that $g_n(x)$ is also even $\forall n$.

Then $g_n \in Q(A)$, $g_n \to g$ in $\mathcal{H} = L^2(\mathbb{R}, dx)$ and

$$q(g_n - g_m, g_n - g_m) = \int x \, |g_n(x) - g_m(x)|^2 \, dx = 0$$

However $g \notin Q(A)$, so q is not closed.

and yet not "self-adjoint" in the following sense.

Th^m 129.1

If q is a closed semi-bded quadratic form, then q is the quadratic form of a unique ~~semi-bded~~ self-adjoint operator.

Proof: Wlog assume $q(\varphi,\varphi)\geq 0$. Then since q is closed and symmetric, $Q(q)$ is a Hilbert space which we denote by $\mathcal{H}_{+1}$, under the inner product

(129.1) $$(\varphi,\psi)_{+1} = q(\varphi,\psi) + (\varphi,\psi)$$

We denote by $\mathcal{H}_{-1}$ the space of bded conjugate linear functionals on $\mathcal{H}_{+1}$. Let j, given by $\psi \overset{j}{\mapsto} (\cdot,\psi)$ be the linear imbedding of $\mathcal{H}$ in $\mathcal{H}_-$. Clearly j is bded because

$$|(j(\psi))(\phi)| = |(\phi,\psi)| \leq \|\phi\|\,\|\psi\| \leq \|\phi\|_{+1}\|\psi\|$$

Also j is 1-1.

Since the identity i embeds $\mathcal{H}_{+1}$ in $\mathcal{H}$ we have a scale of spaces

$$\mathcal{H}_{+1} \xrightarrow{i} \mathcal{H} \xrightarrow{i} \mathcal{H}_{-1}$$

By the Riesz Lemma applied to $\mathcal{H}_{+1}$, the map

$\hat{B}: \Phi \to \hat{B}\Phi$,

(130.1) $$\hat{B}\Phi(\psi) = (\psi, \Phi)_{+1} \qquad , \psi \in \mathcal{H}_{+1}$$

is an isometric isomorphism of $\mathcal{H}_{+1}$ onto $\mathcal{H}_{-1}$

Let $D(B) = \{\psi \in \mathcal{H}_{+1} : \hat{B}\psi \in \operatorname{ran} j\}$. Define

B on $D(B)$ by $B = j^{-1}\hat{B}$, $B: D(B) \to \mathcal{H}$

$\uparrow$

$\subset \mathcal{H}_{+}$

$$\mathcal{H} \supset \mathcal{H}_{+1} \xrightarrow{\hat{B}} \mathcal{H}_{-1} \xleftarrow{j} \mathcal{H}$$

First we prove that the range of j is dense in $\mathcal{H}_{-1}$. If not, there would $\exists\ \lambda \in \mathcal{H}_{-1}^{*}$ st $\lambda \neq 0$ and $\lambda(j(\psi)) = 0$ $\forall \psi \in \mathcal{H}$. Then $\Phi \mapsto \lambda(\hat{B}\Phi)$ is a bdd linear functional on $\mathcal{H}_{+1}$ and hence by the Riesz lemma $\lambda(\hat{B}\Phi) = (\phi_\lambda, \Phi)_{+1}$ for some $\phi_\lambda \in \mathcal{H}_{+1}$. But then by (130.1), $\lambda(\hat{B}\Phi) = (\hat{B}\Phi)(\phi_\lambda)$

In particular for $\hat{B}\Phi = j(\psi)$,

$$0 = \lambda(j(\psi)) = [j(\psi)](\phi_\lambda) = (\phi_\lambda, \psi)_{\mathcal{H}}$$

$\forall \psi \in \mathcal{H}$. Hence $\phi_\lambda = 0$ and so $\lambda = 0$, as $\hat{B}$ $\boxed{\text{is surjective: this is a contradiction}}$. Therefore Ran j is dense in $\mathcal{H}_{-1}$. As $\hat{B}$ is an isometric isomorphism of $\mathcal{H}_{+1}$ onto $\mathcal{H}_{-1}$, it follows that $D(B) \equiv \hat{B}^{-1}(\operatorname{ran} j)$ is dense in $\mathcal{H}_{+1}$. It follows that $D(B)$ is dense in $\mathcal{H}$.

Indeed if $\varphi \in \mathcal{H}$ and $\varepsilon > 0$ is given, then by definition, $\exists\ \varphi_{+1} \in \mathcal{H}_{+1}$ st $\|\varphi - \varphi_{+1}\|_{\mathcal{H}} < \varepsilon/2$. But then $\exists$ $\varphi_B \in D(B)$ st $\|\varphi_B - \varphi_{+1}\|_{\mathcal{H}} \leq \|\varphi_B - \varphi_{+1}\|_{+1} < \varepsilon/2$

Hence $\|\varphi - \varphi_B\|_{\mathcal{H}} < \varepsilon$.

Now suppose $\varphi, \psi \in D(B) \subset \mathcal{H}_{+1}$. Now

(131.1) $$(\varphi, \psi)_{+1} = (\hat{B}\psi)(\varphi) = (j \circ B\psi)(\varphi) = (\varphi, B\psi).$$

Similarly

$$(\psi, \varphi)_{+1} = (\psi, B\varphi) \Rightarrow (\varphi, \psi)_{+1} = \overline{(\psi, \varphi)_{+1}} = \overline{(\psi, B\varphi)} = (B\varphi, \psi)$$

Thus $(\varphi, B\psi) = (B\varphi, \psi)$ ie B is a densely defined symmetric oper.

Now we show that B is s.adjoint. Let

$C = \hat{B}^{-1} j$, C takes $\mathcal{H}$ into $D(B) \subset \mathcal{H}_+ \subset \mathcal{H}$ and is everywhere defined & symmetric.

Indeed, we clearly have $BC = 1_{\mathcal{H}}$ and $CB = I_{D(B)}$.

Thus if $f, g \in \mathcal{H}$, then $f = B\varphi$, $g = B\psi$ for suitable

φ, ψ in $D(B) \subset \mathcal{H}$ (in fact, $\varphi = Cf$ and $\psi = Cg$). Thus

$$(f, Cg) = (f, CB\psi) = (f, \psi) = (B\varphi, \psi)$$

and

$$(Cf, g) = (CB\varphi, g) = (\varphi, B\psi)$$

and so C is symmetric as B is symmetric. Thus by Hellinger-Toeplitz, C is a bdd, s.adj. operator. A simple application of the spectral theorem in mult. op. form then shows that $C^{-1} : \operatorname{Ran} C \to \mathcal{H}$ is s.adj. But $C^{-1} = B$. and so B is s.adj.

Insert → 132+

We now define $A = B - I$. Then A is also s.adj on $D(A) = D(B)$ and by (131.1) for $\varphi, \psi \in D(A)$,

$$(\varphi, A\psi) = q(\varphi, \psi)$$

Since $D(A) = D(B)$ is $\|\cdot\|_{+1}$-dense in $\mathcal{H}_{+1}$, we then have

Insert on p132

Alternatively it follows abstractly from the fact that

- B is symmetric

and

- $\exists$ a bded self-adj operator $C : \mathcal{H} \to D(B)$ st

$$BC = 1_{\mathcal{H}} \quad , \quad CB = 1_{D(B)}$$

that B is s.adj. Indeed, (as C is s.adj), if $g \in \mathcal{H}$ we can find $f \in \mathcal{H}$ st $(1 + iC) f = Cg$. As Cf, $Cg \in D(B) \Rightarrow f = Cg - iCf \in D(B)$. Hence

$$Bf + iBCf = BCg \quad \text{i.e.} \quad Bf + if = g$$

Thus $\mathrm{Ran}\,(B+i) = \mathcal{H}$. Similarly $\mathrm{Ran}\,(B-i) = \mathcal{H}$. By the basic s.adjointness Th^m, this $\Rightarrow B$ is s.adj.

(133.1) $q(\varphi, \psi) = (\varphi, A\psi) \quad \forall \varphi \in Q(q)$.

Uniqueness of A follows from the following Claim.

Lecture 11: We now describe $D(A)$ in a more concrete fashion.

We have:

(133.2) Claim: $D(A) = \{\varphi \in Q(q) : q(\psi, \varphi) = (\psi, x_\varphi)$ for some x_φ and $\forall \psi \in Q(q)\}$

Furthermore, if $\varphi \in D(A)$, then $A\varphi = x_\varphi$

Proof: From (131.1) we have for $\varphi, \psi \in D(B) = D(A)$

(133.3) $$q(\psi, \varphi) = (\psi, A\varphi)$$

But $D(A)$ is $\|\cdot\|_{+1}$-dense in $\mathcal{H}_{+1}$ hence (134.1) holds $\forall$ $\psi \in Q(q)$ with $X_\varphi = A\varphi$. Thus $D(A) \subset$ RHS of the Claim

Conversely suppose for some x_φ

$$q(\psi, \varphi) = (\psi, x_\varphi) \qquad \forall \psi \in Q(q)$$

ie.

$$(\psi, \varphi)_{+1} = (\psi, x_\varphi) + (\psi, \varphi) = (\psi, x_\varphi + \varphi)$$

But $(\psi, \varphi)_{+1} = [\hat{B}\varphi](\psi)$ $\therefore$ $[\hat{B}\varphi](\psi) = (\psi, x_\varphi + \varphi) = [j(x_\varphi + \varphi)](\psi)$

$\forall \psi \in Q(q)$. Hence $\varphi = \hat{B}^{-1} \circ j\,(x_\varphi + \varphi) \in D(B) = D(A)$. Thus RHS $\subset D(A)$

Also $B\varphi = j^{-1}\hat{B}\hat{B}^{-1} \circ j(x_\varphi + \varphi) = x_\varphi + \varphi$ ie $A\varphi = x_\varphi$.

~~Thus: $(\psi, \varphi)_{+1} = (\psi, B\varphi)$ or $q(\psi, \varphi) = (\psi, A\varphi)$ $\forall \psi \in \mathcal{H}_{+1}$.~~

This proves the claim.

Remark Clearly it is enough to know that $q(\psi, \varphi) = (\psi, X_\varphi)$ $\forall \psi$ in a core for q, to conclude that $q(\psi, \varphi) = (\psi, x_\varphi)$ $\forall \psi \in Q(q)$