and

\[ u_+(0) + X_u u_-(0) = 0, \quad u_+(1) + X_u u_-(1) = 0 \]
\[ u_+(0) + X_u u_-(0) = 0, \quad u_+(1) + X_u u_-(1) = 0 \]

But \( u_+(1) = u_+(0) \) and \( u_-(1) = -u_-(0) \) so it is enough to verify

\[ u_+(0) + X_u u_-(0) = (4 + e^{\lambda t}) + X_u (1 + e^{-\lambda t}) = 0 \]
\[ u_+(0) + X_u u_-(0) = (1 - e^{\lambda t}) + X_u (1 - e^{-\lambda t}) = 0 \]

As \( \lambda = \lambda_+ \), we clearly have

\[ |X_u| = |X_u| = 1 \]

This completes the construction of \( U \) with the desired properties.

Exercise

Find \( U \) for \( T_{a,b}, \quad -\infty < a, b < \infty \) and \( T_{a,e} \).

Lecture 10

Another very elegant application of von Neumann's theorem concerns the Hamburger moment problem.

Theorem: The Hamburger moment problem is to determine conditions on a sequence of real numbers \( \sigma_n \) \( n = 0, 1, 2, \ldots \) such that

\[ a_n = \int x^n \, d\sigma(x), \quad n = 0, 1, 2, \ldots \]

The \( \sigma \)'s are called the moments of the measure \( \sigma \).

The Hamburger moment problem is to determine conditions on a sequence of real numbers \( \sigma_n \) \( n = 0, 1, 2, \ldots \) so that

\[ a_n = \int x^n \, d\sigma(x), \quad n = 0, 1, 2, \ldots \]
Theorem 120.1

A sequence of real $\beta_i$'s $\{\alpha_i\}$ are the moments of a pos. meas. on $\mathbb{R} \subset \mathbb{R}$ and all $\beta_0, \beta_1, \ldots, \beta_n \in \mathbb{C}$

(120.2) \[ \sum_{n, m = 0}^{\infty} \beta_n \beta_m \alpha_{n + m} > 0 \]

Proof: First suppose that $\rho$ is a pos. meas and (119.1) holds. Then

\[ \sum_{n, m = 0}^{\infty} \beta_n \beta_m \alpha_{n + m} = \int_{-\infty}^{\infty} \left( \sum_{n = 0}^{\infty} \beta_n x^n \right)^2 d\rho(x) > 0 \]

Conversely, suppose that (120.2) holds, let $\rho$ be the $\alpha$-poly's on $\mathbb{R}$ with complex coefficients and define the sesquilinear form on $\rho$ by

\[ \left( \sum_{n = 0}^{N} \beta_n x^n, \sum_{m = 0}^{M} a_m x^m \right) = \sum_{m = 0}^{M} \sum_{n = 0}^{N} a_{n + m} \beta_n \alpha_m \]
Big (120.2) The term is non-negative. The standard argument shows that we must have the Schwartz inequality

$$|<\pi, \sigma>| \leq \left( \pi, \pi \right)^{\frac{1}{2}} \left( \sigma, \sigma \right)^{\frac{1}{2}}$$

for all $\pi, \sigma \in \mathcal{D}$.

Let $Q = \{ \pi \in \mathcal{D} : (\pi, \pi) = 0 \}$.

Necessarily $Q$ is a linear subspace of $\mathcal{D}$ for

if $\pi, \sigma \in Q$, then

$$<a\pi + b\sigma, a\pi + b\sigma>$$

$$= |a|^2 <\pi, \pi> + |b|^2 <\sigma, \sigma>$$

$$+ \overline{a}b <\pi, \sigma> + a\overline{b} <\sigma, \pi>$$

$$= |a|^2 <\pi, \pi> + |b|^2 <\sigma, \sigma>$$

$$= 0$$,

as $$(121.1)$$.
Let $H$ be the Hilbert space obtained by completing $P/\mathcal{Q}$ in the inner product

\[(\tilde{\pi}, \tilde{\sigma}) = (\pi, \sigma)\]

for $\tilde{\pi} = \pi + \mathcal{Q}$, $\tilde{\sigma} = \sigma + \mathcal{Q}$. Note that, again by (121.11), $(\tilde{\pi}, \tilde{\sigma})$ is a well-defined inner product on $P/\mathcal{Q}$: $(\tilde{\pi}, \tilde{\pi}) \geq 0 \quad \forall \tilde{\pi} \in P/\mathcal{Q}$ and

$(\tilde{\pi}, \tilde{\pi}) = 0 \iff \tilde{\pi} = 0$.

Consider the map $A : P \to P$ defined by

$$A : \sum_{n=0}^{N} \beta_n x^n \mapsto \sum_{n=0}^{N} \beta_n x^{n+1}.$$ 

It is easy to see that $A$ is sym. and $A : \mathcal{Q} \to \mathcal{Q}$ again by the Schwartz inequality

$$|\langle A\mathcal{Q}, A\mathcal{Q} \rangle| = |\langle A^2 \mathcal{Q}, \mathcal{Q} \rangle| \leq (A^2 \mathcal{Q}, \mathcal{Q}) (\mathcal{Q}, \mathcal{Q})^{1/2}.$$ 

So $\mathcal{Q} \in \mathcal{Q} = A \mathcal{Q}$. Thus $A$ descends to a sym. oper $\tilde{A}$ on $\tilde{H}$ with domain $P/\mathcal{Q}$. 

\( \tilde{A}(\pi + \alpha) = \hat{A}\pi + \alpha. \)

If \( C \) denotes the usual complex conj on \( P \), then it is easy to see that \( C \) drops down to a conjg. \( \tilde{C} \) on \( \mathbb{H} \) and \( \tilde{A} \tilde{C} = \tilde{C} \tilde{A}. \) Thus \( \tilde{A} \) has some s. adj. extension, call it \( \tilde{A}. \) Let \( \tilde{p} \) be the spec. meas. for the vector \( \tilde{1} = 1 + \alpha \) in \( P/\alpha \subset \mathbb{H}. \) Then

\[
\int x^n \phi(x) \, dx = (\tilde{1}, \tilde{A}^n \tilde{1})
\]

\[
= (\tilde{1}, \tilde{A}^n \tilde{1})
\]

\[
= (1, A^n 1)
\]

\[
= a_n. \quad \text{QED.}
\]

In general \( \tilde{p} \) is not unique. However (see RS [205-206], if \( |1 \alpha| \leq c D^n n! \gamma n \), for some \( c, D > 0 \), then \( \tilde{p} \) is unique.
A very important aspect of Hilbert space theory is the relationship between operators and quadratic forms.

(see RS Vol I p.76 et seq.)

One consequence of the Riesz representation theorem is that there is a 1-1 correspondence between bounded quadratic forms and bounded operators. That is, any sesquilinear map \( q : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \) which satisfies

\[
|q(u, v)| \leq M \|u\| \|v\| \tag{124.1}
\]

is of the form \( q(u, v) = \langle u, A v \rangle \) for some bounded operator \( A \).

The situation is more complicated if one removes the boundedness restriction.

Definition 124.2: A quadratic form is a map

\[
q : \mathcal{H} \times \mathcal{H} \to \mathbb{C}
\]

where \( \mathcal{H} \) is a dense linear subspace of \( \mathcal{H} \) called the form.
domain, such that $d(z, y)$ is conjugate linear and $d(y, y)$ is linear for $y, z \in \mathfrak{m}$. If $d(y, y) = d(y, y)$, we say that $d$ is symmetric. If $d(y, y) > 0 \, \forall y \neq 0$, $d$ is called positive and if $d(y, y) > -M \|y\|^2$ for some $M$, we say that $d$ is semi-bounded. Note that by polarization, if $\mathfrak{m}$ is complex, then $d$ semi-bounded $\iff$ $d$ symmetric.

**Example:** Let $\mathfrak{m} = L^2(\mathbb{R})$ and $Q(y) = \mathcal{C}_0(\mathbb{R})$ with $d(x, y) = \overline{y(0)} \, y(0)$. One could formally write $d(x, y) = (x, Ay)$

where $A : y \rightarrow \delta(x) \, y(x)$, $\delta = \text{delta function at } 0$.

Since mult. by $\delta(x)$ is not an operator, $d$ is an example of a quadratic form not likely to be associated with an operator.
Example 2. Let $A = A^x$ on $\mathbb{H}$: by the spectral theorem, $A$

is isomorphic to multi. by $x$ on $\bigoplus_{n=1}^{N} L^2(\mathbb{R}, q_{\mu_n})$.

Let

$$Q(a) = \sum_{n=1}^{N} x \left\langle x, a_n \right\rangle q_{\mu_n} \left( x \right) < \infty$$

and for $\psi \in Q(a)$

$$Q(a, \psi) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \psi(x) x q_{\mu_n} \left( x \right) \, dx,$$

We call $Q$ the \textit{quadratic form} associated with $A$ and write

$Q(a) = Q(A)$; $Q(A)$ is called the \textit{form domain of the operator} $A$. For $\psi, \psi \in Q(A)$ we will often write

$$Q(a, \psi) = \left( \psi, A \psi \right)$$

even though $A$ does not make sense on all $\psi \in Q(A)$.

$Q(A)$ is in some sense the \textit{largest domain} on which $Q$ is defined. Note that we always have $D(A) \subset Q(A)$.

We need to extend the notion of "closed"
from operators to forms. Recall that A is closed \( \Rightarrow \)
its graph is closed which is the same as saying \( \text{D}(A) \)
closed under the norm \( \|u\|_A = \| Au\| + \|u\| \). Analogously we

**Definition:** Let \( q \) be a semi-bounded quadratic form, \( q(4,4) = -14 \|u\| \). We say \( q \) is closed if \( \mathcal{Q}(q) \) is complete under

the norm

\[
\|u\|_1^2 = \sqrt[q(4,4) + (4n+1)]{\|u\|_1^4}.
\]

**Exercise (12.5.1):** Show that \( \|u\|_1 \) is indeed a norm

**Exercise (12.5.2):** Show that

\[
q \text{ is closed } \iff \left[ q_{n} \in Q(A), \; q_{n} \to q, \; d(4n - 4m, 4n - 4m) \to 0 \right]
\]

**Exercise (12.5.3):** Show that the quadratic form associated with

a semi-bounded self-adjoint operator \( A \) (see exercise above) is
closed. Moreover if \( \text{D}(A) \) is a core for \( A \), ie \( A^* D = A \)

then \( D \) is a core for \( q \) if \( q \) if \( q \in \mathcal{Q}(A) \), then \( \exists \; q_{n} \in D \)
such that \( \|q_{n} - q\|_1 \to 0 \).

**Exercise:** \( D(A) \) is a core for \( q \).
Consider Example 1, \( q(\varphi, \psi) = \varphi(0) \psi(0) \), \( \varphi, \psi \in C^0(\mathbb{R}) \).

Suppose \( q \) has a closed extension with domain \( \tilde{D} \). Then

if \( \{ \varphi_n \} \subset \tilde{D} \), \( \varphi_n \rightarrow \varphi \) and \( q(\varphi_n - \varphi_m, \varphi_n - \varphi_m) \rightarrow 0 \) then

\( \varphi \in \tilde{D} \) and \( q(\varphi_n - \varphi, \varphi_n - \varphi) \rightarrow 0 \). Let \( \gamma_n \in C^0(\mathbb{R}) \subset \tilde{D} \) be defined as follows:

![Diagram of \( \gamma_n \) function]

Then \( \gamma_n \rightarrow 0 \) in \( \tilde{D} \) and \( q(\gamma_n - \gamma_m, \gamma_n - \gamma_m) = 0 \). But

\( \gamma \in C^0(\mathbb{R}) \subset \tilde{D} \) and

\[ q(\gamma_n - \gamma, \gamma_n - \gamma) = q(\gamma_n, \gamma_n) = 1 \not\rightarrow 0 \]

Thus \( q \) has no closed extension; in particular, \( \tilde{D} \) cannot be a semi-bounded adj. op. A st. \( q(\varphi, \psi) = (\varphi, A\psi) \) for

\( \varphi, \psi \in C^0(\mathbb{R}) \subset D(A) \).

The deep fact about semi-bounded quadratic forms is that

unlike the case for operators, they cannot be closed & symmetric,
Remark In general we do not know that the quadratic form associated with a s. adj. operator $A$ is closed if $A$ is not semi-bounded. For suppose $A$ is multiplication by $x$ in $L^2(\mathbb{R}, dx)$. Then $Q(1) = \frac{1}{2}$.

Let $f \in L^2(\mathbb{R}, dx)$, $\|f\|_2 = \left(\int |f|^2 \, dx\right)^{\frac{1}{2}} < \infty$. Now let $g \in L^2(\mathbb{R}, dx)$, $g \neq 0$. Let $g_n(x) = \chi_{(-n,n)}(x) g(x)$, where $\chi_{(-n,n)}$ is the characteristic function of the set $(-n,n)$. Note that $g_n(x)$ is also even.

Then $g_n \in Q(A)$, $g_n \rightharpoonup g$ in $H = L^2(\mathbb{R}, dx)$ and

$$
\int \phi(x) g_n(x) \, dx = 0
$$

for all $\phi \in L^2(\mathbb{R}, dx)$. However $g \notin Q(A)$, so $Q$ is not closed.
and yet not "self-adjoint" in the following sense.

**Theorem 129.1**

If $q$ is a closed semi-bounded quadratic form, then $q$ is the quadratic form of a unique self-adjoint operator.

**Proof:** Wlog assume $q(4, 4) > 0$. Then since $q$ is closed and symmetric, $q$ is an Hilbert space which we denote by $H_1$, under the inner product

\[(129.1): \quad (\ell, 4)_{H_1} = q(\ell, 4) + (4, 4)\]

We denote by $H_1^*$ the space of bounded conjugate linear functionals on $H_1$. Let $j$, given by $4 \mapsto (\ell, 4)$, be the linear imbedding of $H$ in $H_1$. Clearly, $j$ is bounded because

\[\|j(\ell)(\phi)\| = \|\ell(\phi, 4)\| = \|\ell\| \|\phi\| \|4\| = \|\phi\|_{H_1} \|\ell\|_{H_1} \|4\|\]

Also $j \circ j = 1$. Since the identity $i$ embeds $H_1$ in $H$ we have a scale
spaces

By the Riesz Lemma applied to $H^+_1$, the map

$\hat{B} : H^+_1 \to H^+_1$,  

$$\hat{B}(u) = (u, \hat{\varphi})_+, \quad u \in H^+_1,$$

is an isometric isomorphism of $H^+_1$ onto $H^+_1$.

Let $D(B) = \{ u \in H^+_1 : \hat{B}u \in \text{ran } j \}$. Define

$B$ on $D(B)$ by $B = j^{-1} \hat{B}$, \quad $B : D(B) \to H^+_1$, $j : H^+_1 \to H$. 

First we prove that the range of $j$ is dense in $H^+_1$. If not, there would be $\lambda \in H^+_1$, $\lambda \neq 0$ and $\lambda(j(u)) = 0$ for some $u$. Now $\lambda \to \lambda(\hat{B} \varphi)$ is a bounded linear functional on $H^+_1$ and hence by the Riesz Lemma $\lambda(\hat{B} \varphi) = (\varphi, \varphi)_+^+$ for some $\varphi \in H^+_1$. But then by (130.1), $\lambda(\hat{B} \varphi) = (\hat{B} \varphi)(\varphi)$. 

In particular for \( \overline{B^*} = \hat{\mathcal{J}}(4) \),

\[
0 = \lambda(\hat{\mathcal{J}}(4)) = \left[ \hat{\mathcal{J}}(4^+) \right] (\psi_\lambda) = (\phi_\lambda, 4)
\]

Hence, \( \phi_\lambda = 0 \) and no \( \lambda = 0 \). Therefore \( \text{Ran} \hat{\mathcal{J}} \) is closed in \( \mathcal{H}_0 \). As \( \hat{B} \) is an isometric isomorphism of \( \mathcal{H}_+ \) onto \( \mathcal{H}_- \), it follows that \( \text{Ran} \hat{B} = \hat{B}^{-1}(\text{Ran} \hat{\mathcal{J}}) \) is closed in \( \mathcal{H}_+ \). It follows that \( \text{Ran} \hat{B} \) is closed in \( \mathcal{H}_+ \).

Indeed if \( |\varphi| \in \mathcal{H}_+ \) and \( \varepsilon > 0 \) is given, then by definition,

\[
\exists \psi_+ \in \mathcal{H}_+ \quad \text{s.t.} \quad \| \varphi - \psi_+ \|_{\mathcal{H}_+} < \varepsilon / 2.
\]

But then \( \exists \psi_{-1} \in \mathcal{H}_- \quad \text{s.t.} \quad \| \psi_{-1} - \psi_+ \|_{\mathcal{H}_-} < \varepsilon / 2 \).

Hence \( \| \psi_{-1} - \psi_\lambda \|_4 < \varepsilon \).

Now suppose \( \varphi, \psi \in \text{Ran} \hat{B} \subset \mathcal{H}_+ \). Now

\[
(\varphi, \psi)_{+1} = \left( \hat{\mathcal{J}}(4) \right)(\psi) = (\mathcal{J}_0 \hat{B}_4)(\psi) = (\psi, \hat{B}_4).
\]

Similarly

\[
(\varphi, \psi)_{+1} = (\varphi, \hat{B}_4) = (\varphi, \psi)_{+1} = (\varphi, \psi)_{+1} = (\varphi, \overline{\psi})(\psi)
\]

Thus \( (\varphi, \hat{B}_4) = (\psi, \varphi, \psi) \) if \( \hat{B} \) is a densely defined symmetric operator.
Now we show that $B$ is s. adjoint. Let

$$C = B^{-1} \mathbf{j},$$

$C$ takes $\mathcal{H}$ into $D(B^*) \mathcal{H} \mathcal{C}$ and is symmetric.

Indeed, we clearly have

$$BC = I \mathcal{H} \quad \text{and} \quad CB = I \mathcal{D}(B).$$

Thus if $f, g \in \mathcal{H}$, then $f = B g$, $g = B f$ for suitable

$$u, \psi \in \mathcal{D}(B) \mathcal{C} \mathcal{H}$$

(Indeed, $\psi = C f$ and $u = C g$). Thus

$$(\psi, C g) = (f, C B u) = (f, u)$$

and

$$(C f, \psi) = (C B g, \psi) = (u, B \psi)$$

and so $C$ is symmetric as $B$ is symmetric. Thus by

Hellinger-Toeplitz, $C$ is a b-d, s. adj. operator. A simple

application of the spectral theorem in mul. op. form

Now shows that $C^{-1} : \text{ran } C \to \mathcal{H}$ is s. adj. But $C^{-1} = B$.

and so $B$ is s. adj.

We now define $A = B - I$. Then $A$ is also s. adj on

$D(A) = D(B)$ and by (131.1) for $u, v \in D(A)$,

$$(u, A v) = d(u, v)$$

Since $D(A) = D(B)$ is $((1 + 1)^{-1} - \text{dense })$ in $\mathcal{H}$, we then have
Alternatively it follows from the fact that

- \( B \) is symmetric

and

- A bounded self-adjoint operator \( C : D(B) \to \mathbb{D}(B) \) s.t.
  \[
  BC = i \mathbf{1}, \quad CB = -\mathbf{1}_B
  \]

That \( B \) is s. adj. Indeed, \( \forall f, g \in \mathbb{H} \), we can find \( f \in \mathbb{H} \) s.t. \( (j + iC)f = Cg \). As \( Cg \in D(B) = \mathbb{H} \), we can find \( f \in \mathbb{H} \) such that \( (j + iC)f = Cg \). Hence

\[
Bf + iBCf = BCg \quad \text{i.e.} \quad Bf + iCf = g
\]

Thus \( \text{Range} \ (B + iC) = \mathbb{H} \). Similarly \( \text{Range} \ (B - iC) = \mathbb{H} \). By the basic adjointness theorem, \( \forall h \in \mathbb{D}(B) \),

\[
1 \Rightarrow B \text{ is s. adj}.
\]
(33.1) \( q(0, q) = (0, Aq) \) \( \forall q \in q(A) \).

Uniqueness of \( A \) follows from the following Claim.

Lecture 11: We now describe \( D(A) \) in a more concrete fashion.

We have:

(33.2) **Claim:** \( D(A) = \{ q \in q(A) : q(4, q) = (4, \chi_q) \text{ for some } \chi_q \in A(q) \} \).

Furthermore, if \( q \in D(A) \), then \( Aq = \chi_q \).

**Proof:** From (33.1) we have for \( 4, 4 \in B(B) = D(A) \):

(33.3) \( q(4, q) = (4, Aq) \).

But \( D(A) \) is \( H_{+1} \)-dense in \( H_{+1} \), hence (33.1) holds for \( 4 \in B(q) \) with \( \chi_q = Aq \). Thus \( D(A) \subset \text{RHS of the Claim} \).

Conversely, suppose for some \( \chi_q \):

\( q(4, q) = (4, \chi_q) \) \( \forall 4 \in q(A) \).

Then:

\( (4, q)_{+1} = (4, \chi_q) + (4, q) = (4, \chi_q + q) \).

But \( (4, q)_{+1} = [B^0 q](4) \text{ and } [B^0 q](4) = (4, \chi_q + q) = [\hat{\mu}(q)](4) \).

Thus:

\( q(4, q) = (4, \chi_q) \).

This proves the Claim.

Remark: Clearly it is enough to know \( q(4, q) = (4, \chi_q) \) \( \forall 4 \) in a core for \( q \), to conclude that \( q(4, q) = (4, \chi_q) \) \( \forall 4 \in q(A) \).