Problem 1. If $Y$ is a proper subspace of a Hilbert space $H$, then there exists $x \in H$ such that $\|x\| = 1$ and $\text{dist}(x, Y) = 1$. Show by example that this result is not true for all Banach spaces $X$.

Problem 2. Let $X$ be a Banach space. Show that if $X'$ is separable, then $X$ is separable. Conversely, show by example that if $X$ is separable, then $X'$ may not be separable.

Problem 3. Let $\{x_n, n \in \mathbb{Z}\}$ be a net of vectors in a Hilbert space $H$, so that $a_{nm} = \langle x_n, x_m \rangle$ is the matrix (in the natural basis) of an operator $A$ on $l_2(\mathbb{Z}, \mathbb{C})$. Prove that

$$\sum_{n=-\infty}^{\infty} |\langle f, x_n \rangle|^2 \leq \|A\| \|f\|_2^2$$

for all $f \in H$.

Problem 4. Let $X$ and $Y$ be Banach spaces. Suppose that for $n \geq 1$, $T_n \in K(X, Y)$ (the compact operators from $X$ to $Y$), and let $T_n \to T$ in operator norm. Show that $T \in K(X, Y)$.
Problem 5. Let \( K(x,y) \) be a continuous function on \([0,1] \times [0,1]\). Define

\[
Tf(x) = \int_0^1 K(x,y) f(y) \, dy
\]

for any integrable function \( f \) on \([0,1]\). Show that for any \( 1 \leq p \leq \infty \), \( T \) is a compact map from \( L^p([0,1], dx) \) to itself.

Problem 6. Let \( P \) and \( Q \) be orthogonal projections onto subspaces \( M \) and \( N \) in a Hilbert space \( H \). Prove that

\[
R = \lim_{n \to \infty} (P Q)^n \chi
\]

exists for all \( x \in H \) and that \( R \) is the orthogonal projection onto \( M \cap N \).

Problem 7. Let \( a, b, c, d \in C \setminus S' \) where \( S' = \{ z : |z| = 1 \} \).

For \( z \in S' \), let

\[
h(z) = \frac{3-a}{(3-b)(3-c)}
\]

Show that the associated Toeplitz operator

\[
T_h f = P_h(h f), \quad f \in H
\]

is Fredholm and compute \( \text{ind } T_h \), \( \dim \ker T_h \) and \( \text{codim } T_h \).
Problem 8

Show that if $A \in \mathbb{B}_1(\mathbb{H})$ and $B \in \mathbb{B}_1(\mathbb{H})$, then

$$\|BA\| \leq \|B\| \|A\|,$$

and

$$\|A-B\| \leq \|B\| \|A\|.$$ 

Theorem 1

Show that if $A$ and $B$ are trace class, if $A \in \mathbb{B}_1(\mathbb{H})$ and $B \in \mathbb{B}_1(\mathbb{H})$, then $A+B \in \mathbb{B}_1(\mathbb{H})$.

Theorem 2

Show that the unit sphere $\{x : \|x\| \leq 1\}$ in a Banach space is compact if and only if $X$ is finite dimensional.