Final Examination Functional Analysis Fall 2015

Prepare the following Problems and Theorems

Problem 1 If Y is a proper subspace of a Hilbert space $\mathcal{H}$, then there exists $x \in \mathcal{H}$ such that

$$\|x\| = 1 \quad \text{and} \quad \operatorname{dist}(x, Y) = 1$$

Show by example that this result is not true for all Banach spaces X.

Problem 2 Let X be a Banach space. Show that if X' is separable, then X is separable. Conversely, show by example that if X is separable, then X' may not be separable.

Problem 3 Let $\{x_n, n \in \mathbb{Z}\}$ be a set of vectors in a Hilbert space $\mathcal{H}$, so that $a_{nm} = |(x_n, x_m)|$ is the matrix (in the natural basis) of an operator A on $\ell_2(-\infty, \infty)$. Prove that

$$\sum_{n=-\infty}^{\infty} |(f, x_n)|^2 \le \|A\| \, \|f\|^2$$

for all $f \in \mathcal{H}$.

Problem 4 Let X and Y be B-spaces. Suppose that for $n \ge 1$, $T_n \in K(X, Y)$, the compact operators from X to Y, and let $T_n \to T$ in operator norm. Show that $T \in K(X, Y)$

Problem 5 Let $K(x,y)$ be a continuous function on $[0,1]\times[0,1]$. Define

$$Tf(x) = \int_0^1 K(x,y) f(y)\,dy$$

for any integrable function f on $[0,1]$. Show that for any $1 \le p \le \infty$, T is a compact map from $L^p([0,1], dx)$ to itself.

Problem 6 Let P and Q be orthogonal projections onto subspaces $\mathcal{M}$ and $\mathcal{N}$ in a Hilbert space $\mathcal{H}$. Prove that

$$Rx = \lim_{n\to\infty} (PQ)^n x$$

exists for all $x \in \mathcal{H}$ and that R is the orthogonal projection onto $\mathcal{M} \wedge \mathcal{N}$.

Problem 7 Let $a, b, c, d \in \mathbb{C} \setminus S^1$ where $S^1 = \{z : |z| = 1\}$.

For $z \in S^1$, let

$$h(z) = \frac{z-a}{(z-b)(z-c)}$$

Show that the associated Toepletiz operator

$$T_h f = P_+(hf), \quad f \in \mathcal{H}_+$$

is Fredholm and compute ind T_h, dim ker T_h and codim T_h.

Problem 8 Show that if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{L}(\mathcal{H})$, then

$$\|BA\|_1 \leq \|B\| \, \|A\|_1$$

and

$$\|AB\|_1 \leq \|B\| \, \|A\|_1$$

Theorem 1 Show that if A and B are trace class, if $A \in \mathcal{B}_1(\mathcal{H})$ and $B \in \mathcal{B}_1(\mathcal{H})$, then $A + B \in \mathcal{B}_1(\mathcal{H})$

Theorem 2 Show that the unit sphere $\{x : \|x\| \leq 1\}$ in a Banach space is compact if and only if X is finite dimensional.