1. Show that the Analytic Fredholm Theorem is equivalent to Riesz-Schauder Theory i.e. in the proof of Fred. Thm I we used the Riesz-Schauder Thm: conversely, apply Riesz-Nudel. Fred. Thm I to \( L(\mathcal{A}) = \mathcal{A}\), to deduce the facts of Riesz-Schauder Theory.

2. Let \( \mathcal{H}^+ = \left\{ \sum_{n=0}^{\infty} an e^{i\theta} : \sum_{n=0}^{\infty} |an|^2 < \infty \right\} \subset L^2(T, d\theta) \) where \( T \) is the unit circle. Let \( P^+ \) denote the orthogonal projection of \( L^2 \) to \( \mathcal{H}^+ \) given by

\[
P^+ \left( \sum_{n=0}^{\infty} an e^{i\theta} \right) = \sum_{n=0}^{\infty} an e^{i\theta}
\]

Consider the Toeplitz operator \( T_h \) from \( \mathcal{H}^+ \to \mathcal{H}^+ \)

\[
T_h f = P^+ h f, \quad f \in \mathcal{H}^+
\]

where \( h = h(\theta) \in L^\infty(T) \).

Show that
(i) $T_n \in L(\mathcal{H})$

(ii) If $h \in C(T^1), h(\theta) \geq c > 0, 0 \leq \theta \leq 2\pi,$ Then

$T_n$ is Fredholm in $\mathcal{H}$

(Hint: Show that $P_+ h P_-\lambda$ is compact in $L^2(0, 2\pi)$ if $h(\theta)$ is continuous. Here $P_\lambda(\sum_{n=-\infty}^{\infty} a_n e^{in\theta}) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta},$ i.e.,

$P_- = 1 - P_+$

(iii) Let $h(\theta) = 1 - \cos \theta$ on $\mathcal{H}.$ Show that $T_n$ is not

Fredholm (Hint: Show that the functions $g(\lambda) = (3 - \lambda)^{-1},$ $a(\lambda)$ are not in ran $T_n$)

3. Let $A \in L(X), X$ a B-space. Prove that the set of $\lambda$ such that $\lambda$ is in $\sigma(A)$ but not an eigenvalue and ran $(\lambda - A)^{-1}$ is closed but not all of $X,$ is an open subset of $\mathbb{C}.$

4. If an operator $A$ maps an $n$-dimensional space $X$ into an $m$-dimensional space $Y,$ then $\text{ind } A = n - m.$
5. Let $X$ be a separable Banach space. By a Schauder basis in $X$, we mean a set $x_n \in X$, $\|x_n\| = 1$, $n = 1, 2, \ldots$, such that for every $x \in X$, there is a unique series
\[
\sum_{n=1}^{\infty} a_n x_n
\]
such that
\[
\lim_{n \to \infty} \|x - \sum_{i=1}^{n} a_i x_i\| = 0.
\]

(1) Show that $a_n = a_n(x)$ is a bounded linear functional of $x$.

(2) Exhibit a Schauder basis for $C[0, 1]$, the space of continuous functions on $[0, 1]$ with sup norm.

Remarks: (1) Not every separable Banach space has a Schauder basis. This is a result of P. Enflo.

(2) Note also that a Schauder basis very different from a Hamel basis.