Problem Set # 7

Fall 2015

Functional Analysis
due: Oct 29, Thursday

1. (a) Let $T \in L(X)$. Prove that $\lim_{n \to \infty} \|T^n\|^\frac{1}{n}$ exists and is equal to $\inf_{n \in \mathbb{N}} \|T^n\|^\frac{1}{n}$ as follows:

   (i) Set $a_n = \log \|T^n\|$ and prove that $a_{m+n} \leq a_m + a_n$ and any positive integer $n$.

   (ii) For a fixed positive integer $m$, set $n = mq + r$ where $q$ and $r$ are positive integers $0 \leq r < m - 1$. Using (i) conclude that

   $$\ell_m \frac{a_n}{n} \leq \frac{a_m}{m}.$$  

   (iii) Prove that $\lim_{n \to \infty} a_n/n = \inf_{n \in \mathbb{N}} a_n/n$ and thus the desired inequality.

(b) Show that $\lim_{n \to \infty} \|T^n\|^\frac{1}{n} = r(T) = \sup_{\lambda \in \sigma(T)} |\lambda|$

$r(T)$ is called the spectral radius of $T$.

(c) Observe that in general $r(T) \leq \|T\|$. Show that

$r(T) = \|T\|$ if $T = T^*$ is self-adjoint, and show by example that $r(T)$ may be $< \|T\|$ if $T$ is not self-adjoint.
(Hint: Show that by Lemma 100.2 \( \|T^*T\| = \|T\|^2 \))

2. Use Thm 10.2 to give another proof of Criterion 102.1

3. Show by example that if \( U \) is a subspace of a
Banach space \( Y \), and \( \dim (Y/U) < \infty \), then \( U \) may not be
closed.

4. Let \( \Phi = l^2 = \{ x = (x_i)_{i=-\infty}^\infty : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty \} \). Let
\( P \) be the projection w.r.t. \( \Phi \) onto \( l^2_+ \), \( i.e. \) for \( x = (x_i)_{i=-\infty}^\infty \),
\( (Px)_i = x_i \) for \( i \geq 0 \) and \( (Px)_i = 0 \) for \( i < 0 \). Let \( Q \)
be the projection onto \( l^2_- \), \( i.e. \) for \( x = (x_i)_{i=-\infty}^\infty \)
\( (Qx)_i = 0 \) for \( i \geq 0 \)
\[ = x_i \] for \( i < 0 \).

Let \( h = (h_i)_{i=-\infty}^{\infty} \in l_1 \), i.e. \( \sum_{i=-\infty}^{\infty} |h_i| < \infty \), and let \( K_h \)
be the operator of convolution by \( h \), i.e.
\[ (K_hx)_i - (hx)_i = \sum_{j=-\infty}^{\infty} h(i-j)x_j \quad -\infty < i < \infty. \]
Show that the operator \( T = PX_nQ \) i.e.

\[
Tf = P(h_0(Q f))
\]

is compact in \( L_2 \). (Hint: show that \( \|T\| \) in addition

\[
\sum_{k=0}^{\infty} k \|h_k\|^4 < \infty, \text{ then } T \text{ is Hilbert–Schmidt}
\]

5. Show that the spectral radius of the Volterra operator

\[
T\psi(x) = \int_0^x \psi(t)\,dt
\]

as a map in \( C([0,1]) \) is equal to 0. What is \( \|T\| \), the norm of \( T \)?

6. Let \( P \) and \( Q \) be orthogonal projections onto subspaces

\( M \) and \( N \) in a Hilbert space \( \mathcal{H} \) i.e. \( P = P^2 = P^* \) and \( Q = Q^2 \)

= \( Q^* \). Prove that \( Rx \in \lim_{n \to \infty} (PQ)^n x \) exists for all \( x \in \mathcal{H} \)

and that \( R \) is the orthogonal projection onto \( M \cap N \).