Problem Set # 7 Functional Analysis

Fall 2015 due : Oct 29, Thursday

1\. (a) Let $T \in \mathcal{L}(X)$. Prove that $\lim_{n\to\infty} \|T_n\|^{\frac{1}{n}}$ exists and is equal to $\inf_n \|T^n\|^{\frac{1}{n}}$ as follows:

(i) Set $a_n = \log \|T^n\|$ and prove that $a_{m+n} \le a_m + a_n$

(ii) For a fixed positive integer m and any positive integer n set $n = mq + r$ where q and r are positive integers $0 \le r \le m-1$. Using (i) conclude that

$$\overline{\lim_n} \frac{a_n}{n} \le \frac{a_m}{m}$$

(iii) Prove that $\lim_{n\to\infty} a_n/n = \inf_n a_n/n$ and thus the desired inequality

(b) Show that $\lim_{n\to\infty} \|T^n\|^{\frac{1}{n}} = r(T) \equiv \sup_{\lambda\in\sigma(T)} |\lambda|$

$r(T)$ is called the <u>spectral</u> <u>radius</u> of T.

(c) Observe that in general $r(T) \le \|T\|$. Show that $r(T) = \|T\|$ if $T = T^*$ is self-adjoint, and show by example that $r(T)$ may be $< \|T\|$ if T is not self-adjoint.

(Hint: Show that by Lemma 100.2 $\|T^*T\| = \|T\|^2$)

2. Use Th^m 96.2 to give another proof of Criterion 102.1

3. Show by example that if U is a subspace of a Banach space Y, and $\dim(Y/U) < \infty$, then U may not be closed.

4. Let $H = l_2 = \{x = \{x_i\}_{i=-\infty}^{\infty} : \sum_{-\infty}^{\infty} |x_i|^2 < \infty\}$, Let P be the projection in H onto l_2^+, ie for $x = \{x_i\}_{-\infty}^{\infty}$, $(Px)_i = x_i$ for $i \geq 0$ and $(Px)_i = 0$ for $i < 0$. Let Q the projection onto l_2^- ie for $x = \{x_i\}_{-\infty}^{\infty}$

$$(Qx)_i = 0 \quad \text{for } i \geq 0$$
$$= x_i \quad \text{for } i < 0.$$

Let $h = \{h_i\}_{-\infty}^{\infty} \in l_1$, ie. $\sum_{-\infty}^{\infty} |h_i| < \infty$, and let $\mathcal{X}_h$ be the operator of convolution by h, ie

$$(\mathcal{X}_h x)_i = (h \circ x)_i = \sum_{j=-\infty}^{\infty} h(i-j)\, x_j \qquad -\infty < i < \infty.$$

Show that the operator $T = P\chi_h Q$ i.e.

$$Tf = P(h\circ(Qf))$$

is compact in l_2. (Hint: show that $\|T\| \le \|h\|$, and if in addition $\sum_{k=0}^{\infty} k|h_k|^2 < \infty$, then T is Hilbert-Schmidt)

5. Show that the spectral radius of the Volterra operator

$$Tf(x) = \int_0^x f(u)\,du$$

as a map in $C[0,1]$ is equal to 0. What is the norm of T?

6. Let P and Q be orthogonal projections onto subspaces M and N in a Hilbert space $\mathcal{H}$ i.e. $P=P^2=P^*$ and $Q=Q^2=Q^*$. Prove that $Rx \equiv \lim_{n\to\infty} (PQ)^n x$ exists for all $x \in \mathcal{H}$ and that R is the orthogonal projection onto $M \cap N$.