Problem Set #6 — Functional Analysis

Fall 2015 — due: Thursday Oct 17

1. Let $\mathcal{H}$ be a Hilbert space. Show that every bounded sequence in $\mathcal{H}$ contains a subsequence which converges weakly to an element of $\mathcal{H}$.

2. Let X be a Banach space. Show that there are no operators $A \in \mathcal{L}(X)$, $B \in \mathcal{L}(X)$ such that

$$[A, B] = AB - BA = i$$

3. Let $H = -d^2/dx^2 + V(x)$ be the Schrödinger operator acting in $L^2(0,1)$ with domain

$$D = \{ f \in L^2(0,1) : f, f' \text{ absolutely continuous}, \\ f(1) = f(0), \quad f'(1) = f'(0) \}$$

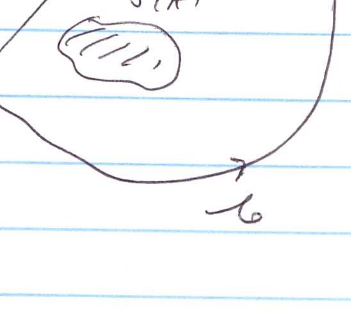
and with $V \in C[0,1]$. Construct another operator $\tilde{H} = -d^2/dx + \tilde{V}$, $\tilde{V} \neq V$, $\tilde{V} \in C[0,1]$ such that $\sigma(\tilde{H}) = \sigma(H)$. (See P. Deift, Duke Math J., 45 (1978), 267–310)

4. Define a <u>functional calculus</u> for an operator $A \in \mathcal{L}(X)$, X a B-space, as follows.

If $f(z)$ is analytic in a neighborhood of $\sigma(A)$, define

(2.1) $$f(A) = \int_{\mathcal{C}} \frac{f(s)}{s-A} \frac{ds}{2\pi i}$$

where $\mathcal{C}$ is any simple closed anti-clockwise contour containing $\sigma(A)$ in its interior



(2.2) Show that if z lies outside $\mathcal{C}$, then

$$\frac{1}{A-z} = \int_{\mathcal{C}} \frac{1}{s-z} \frac{1}{s-A} \frac{ds}{2\pi i}$$

(2.3) If A is a 2×2 matrix, compute e^A by setting $f(z) = e^z$ in (2.1)

(2.4) Prove the Cayley-Hamilton Theorem for $n \times n$ matrices A using (2.1): i.e. if $p(z) = \det(z - A) = z^n + a_1 z^{n-1} + \cdots a_n$, then

$$p(A) = \det(z-A)\Big|_{z=A} = A^n + a_1 A^{n-1} + \cdots + a_n = 0$$

5. Show that every linear space X can be normed.

6. Consider the collection X of all functions on $-\infty < t < \infty$ representable in the form

$$x(t) = \sum_{k=1}^{n} a_k e^{i\lambda_k t}$$

for arbitrary n, real #'s $\lambda_1, \ldots, \lambda_n$, and complex coefficients $a_1, \ldots, a_n$. For $x, y \in X$, define

(3.1) $$(x, y) \equiv \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \overline{x(t)}\, y(t)\, dt.$$

(3.2) Show that with (3.1) X becomes an inner product space.

(3.2) Let $\overline{X}$ be the completion of X with respect to $(\cdot,\cdot)$. Show that the (uncountable) set $\{e^{i\lambda t} : \lambda \in \mathbb{R}\}$ forms an orthonormal basis for the Hilbert space $\overline{X}$.

In particular $\overline{X}$ is not separable.

7. Prove Bessel's inequality, viz., if $\{x_\alpha\}_{\alpha \in A}$ is an orthonormal set in a Hilbert space $\mathcal{H}$, then for any $x \in \mathcal{H}$,

(4.1) $$\|x\|^2 \geq \sum_{\alpha \in A} |(x_\alpha, x)|^2$$

In particular, for any given x, only a countable # of the Fourier coefficients $\{(x_\alpha, x)\}$ are non-zero. The countable set will in general depend on x.