# Functional Analysis Fall 2015 — Problem Set #2

## Due : Thursday Sept. 23

1. Let $X$ be a Banach space. Show that if $X'$ is separable, then $X$ is separable. Conversely, show by example that if $X$ is separable, then $X'$ may not be separable.

2. Show that if $X$ is not finite dimensional, then there exist unbounded linear functionals on $X$

3. Show that $l_\infty = \{x = (x_1, x_2, \dots) : \sup_i |x_i| < \infty\}$
$$\|x\|_\infty = \sup_i |x_i|$$
is not separable.

4. Let $T \in \mathcal{L}(X, Y)$, $X$ and $Y$ Banach spaces.

Show that the following are equivalent :

(a) $\operatorname{ran} T$ is closed

(b) $\exists\, 0 < c < \infty$ st $c\,\|[x]\| \leq \|[T][x]\|$ for all $[x] \in X/\ker T$

(c) $\exists\, 0 < c < \infty$ with the following property : for any $x \in X$, $\exists\, u \in \ker T$ st $c\,\|x + u\| \leq \|T(x+u)\|$

(d) $\exists\, 0 < c < \infty$ with the following property: for any $y \in \operatorname{Ran} T$, $\exists\, x \in X$ st $y = Tx$ and $c\|x\| \le \|y\|$

5. (Hellinger and Toeplitz Theorem)

Let $T$ be a linear operator, $T : \mathcal{H} \to \mathcal{H}$ for some Hilbert space $(\mathcal{H}, (\cdot,\cdot))$. Suppose that $T$ is symmetric i.e.

$$(u, Tv) = (Tu, v) \qquad \forall u, v \in \mathcal{H}.$$

Then $T$ is bounded. Said differently, every symmetric operator that is everywhere defined, i.e. $\operatorname{Dom} T = \mathcal{H}$, is bounded.

6 By the Hahn - Banach Theorem, every Banach space has many continuous linear functions. Metric spaces, even linear metric spaces, may not have <u>any</u> continuous linear functionals other than the zero functional

(Following Royden p244 3rd edition, Real Analysis).

Let $X$ be the vector space of all measurable real valued functions on $[0,1]$ with addition and scalar multiplication in the usual way. Define

$$\sigma(x) = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\, dt \quad , \quad x = x(t) \in X$$

(a) Show that $\sigma(x+y) \le \sigma(x) + \sigma(y)$ and hence

$$\rho(x,y) \equiv \sigma(x-y)$$

defines a metric on $X$.

(b) Show that $(X, \rho)$ is complete

(c) Show that addition is a continuous mapping $X \times X \to X$.

(d) Show that multiplication is a continuous mapping $\mathbb{R} \times X \to X$

(e) Show that the set of step functions is dense in $X$

(f) Show that there is no non-zero continuous linear functional on $X$

(Hint: Suppose $f$ is a cont. lin. func. on $X$. Show that there is a positive integer $n$ st $f(x) \equiv 0$ whenever $x$ is the characteristic function of an interval of length less than $\frac{1}{n}$. Hence $f(x) = 0$ for all step functions.)