1. Let $X$ be a Banach space. Show that if $X'$ is separable, then $X$ is separable. Conversely, show by example that if $X$ is separable, then $X'$ may not be separable.

2. Show that if $X$ is not finite dimensional, then there exist unbounded linear functionals on $X$.

3. Show that $l_\infty = \{x = (x_1, x_2, \cdots) : \sup |x_i| < \infty \}$
   
   \[ \|x\|_\infty = \sup |x_i| \]

   is not separable.

4. Let $T \in \mathcal{L}(X,Y)$, $X$ and $Y$ Banach spaces. Show that the following are equivalent:

   (a) $\text{ran } T$ is closed

   (b) \[ \exists c < \infty \text{ so that } \|T[x]\| \leq c \|x\| \text{ for all } x \in X/\ker T \]

   (c) \[ \exists c < \infty \text{ with the following property: for any } x \in X, \text{ if } x \in \ker T \text{ st } \|x\| \leq c \|T(x+u)\| \]
(d) \( f : C \to \mathbb{C} \) with the following property: for any \( y \in \text{Ran } T \), \( \exists x \in X \) s.t. \( y = Tx \) and \( \|x\| \leq M \|y\| \).

5. (Hellinger and Toeplitz Theorem)

Let \( T \) be a linear operator, \( T : \mathcal{H} \to \mathcal{H} \) for some Hilbert space \((\mathcal{H}, \langle \cdot, \cdot \rangle)\). Suppose that \( T \) is symmetric, i.e.,

\[
\langle u, Tv \rangle = \langle Tu, v \rangle \quad \forall u, v \in \mathcal{H}
\]

Then \( T \) is bounded. Said differently, every symmetric operator that is everywhere defined, i.e., \( \text{Dom } T = \mathcal{H} \),
is bounded.

6. By the Hahn–Banach Theorem, every Banach space has many continuous linear functionals. Metric spaces, even linear metric spaces, may not have any continuous linear functionals other than the zero functional.
(Following Royden p.244 3rd edition, Real Analysis).

Let $X$ be the vector space of all measurable real valued functions on $[0, 1]$ with addition and multiplication in the usual way. Define

$$\sigma(x) = \int_0^1 \frac{|x(t)|}{1 + |x(t)|} \, dt, \quad x = x(t) \in X$$

(a) Show that $\sigma(x+y) \leq \sigma(x) + \sigma(y)$ and hence

$$\rho(x, y) = \sigma(x-y)$$

defines a metric on $X$.

(b) Show that $(X, \rho)$ is complete.

(c) Show that addition is a continuous mapping $X \times X \to X$.

(d) Show that multiplication is a continuous mapping $\mathbb{R} \times X \to X$.

(e) Show that the set of step functions is dense in $X$.

(f) Show that there is no non-zero continuous linear functional on $X$.

(Hint: Suppose $f$ is a cont. lin. func. on $X$. Show that there is a positive integer $n$ such that $f(x) = 0$ whenever $x$ is the characteristic function of an interval of length less than $\frac{1}{n}$. Hence $f(x) = 0$ for all step functions.)