Problem Set # 11

Fall 2015

Functional Analysis

due: Dec 3, 2015
↑
Thursday

1. Prove the "square root lemma", Th^m 156.1

2. Prove Th^m 157.3 : $A \in \mathcal{L}(H) \Rightarrow A = U|A|$ where U is a partial isometry and $|A| = \sqrt{A^* A}$

3. (a) Prove that if A is normal i.e. $AA^* = A^*A$, then $\|A\| = \sup_{\lambda \in \sigma(A)} |\lambda| = r(A)$, the spectral radius of A.

(b) Prove that if $P(x)$ is any polynomial and A is a normal operator, then $\|P(A)\| = \sup_{\lambda \in \sigma(A)} |P(\lambda)|$

4. Let $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that it is false that

$$|(\sigma_3 + I) + (\sigma_1 - I)| \leq |\sigma_3 + I| + |\sigma_1 - I|$$

where $|\cdot|$ denotes the modulus ie $|A| = \sqrt{A^* A}$ for any bdd operator A. Thus $|\cdot|$ is <u>not</u> a norm.

5. (a) Prove that if $A_n \geq 0$, $A_n \to A$ in norm, Then $\sqrt{A_n} \to \sqrt{A}$ in norm

(b) Suppose $A_n \geq 0$, $A \geq 0$ and $A_n \to A$ strongly, i.e. $A_n x \to Ax$ for each x, as $n \to \infty$. Then $\sqrt{A_n} \to \sqrt{A}$ strongly.

6. (a) Let $A_n \to A$ in norm. Prove That $|A_n| \to |A|$ in norm.

(b) Suppose $A_n \to A$ and $A_n^* \to A^*$ strongly. Prove That $|A_n| \to |A|$ strongly

(c) Find an example which shows that $|\cdot|$ is not weakly continuous in $\mathcal{L}(\mathcal{H})$ i.e. $(x, A_n y) \to (x, Ay)$ for each $x, y \in \mathcal{H}$, does <u>not</u> imply $(x, |A_n| y) \to (x, |A| y)$ as $n \to \infty$, $\forall x, y \in \mathcal{H}$.