$= \overline{\operatorname{ran} T}$. Let $y'$ be an extension of $l$ to $Y$.

Then for all $x \in X$, $\langle T'y', x\rangle = \langle y', Tx\rangle$

$= l(Tx) = \langle x', x\rangle$, and so $T'y' = x'$ i.e. $x' \in \operatorname{ran} T'$.

This completes the proof of Banach's theorem (45.3).

Lecture 5

Remark 57.1 Suppose $T \in \mathcal{L}(X, Y)$, and hence $T' \in \mathcal{L}(Y', X')$, has closed range. It is interesting to apply (46.1) to $T'$. We have

(57.2) $\operatorname{ran} T' = \{x' \in X' : \langle x'', x'\rangle = 0 \;\; \forall\, x'' \in \ker T''\}$

But (46.2) also applies, so we also have

(57.3) $\operatorname{ran} T' = \{x' \in X' : \langle x', x\rangle = 0 \;\; \forall\, x \in \ker T\}$

Using the injection $\varphi : X \to X''$ sending $x \mapsto x'' = \varphi(x)$

$x''(x') = x'(x)$ (see (26.3)), we see that

$$\langle x'', x'\rangle = 0$$

for all $x'' \in \ker T''$, if and only if

$$\langle x'', x'\rangle \left(= x'(x)\right) = 0$$

for all $x'' \in M \equiv \ker T'' \cap \operatorname{ran}\varphi$. But in general $M \subsetneq \ker T''$, and so the conclusion is somewhat surprising.

Let $Y$ be a closed linear subspace of a Banach space $X$. We say that closed linear subspace $Y'$ of $X$ is a <u>complement</u> of $Y$ in $X$ if $X$ is a <u>direct sum</u> of $Y$ and $Y'$, $X = Y \oplus Y'$, i.e., any $x \in X$ can be expressed as a <u>unique</u> sum

(58.1) $$x = u + v \quad \text{where } u \in Y \text{ and } v \in Y'$$

Note that uniqueness is equivalent to $Y \cap Y' = \{0\}$.

Complements $Y'$ of $Y$, if they exist, are not unique. For example if $X = \mathbb{R}^2$ and $Y = \{\langle x, 0\rangle : x \in \mathbb{R}\}$, then $Y' = \{\langle 0, y\rangle : y \in \mathbb{R}\}$ and $Y' = \{\langle y, y\rangle : y \in \mathbb{R}\}$ are both complements of $Y$ in $X$. (see also Prob. Set #4).

By Theorem 13.2, every closed subspace $Y$ of a Hilbert space $\mathcal{H}$ has a complement, viz $Y' = Y^\perp$. But in general a Banach space may have subspaces that cannot be complemented. For example in [R.J. Phillips, On linear transformations, Transactions of Amer. Math. Society, 48 (1950), 516-554], Philips showed that $c_0 = \{x = \{x_1, x_2, \ldots\} : \lim_{n\to\infty} x_i = 0\}$ is a subspace in $l^\infty$ that cannot be complemented (see Problem set #4).

A linear operator $P : X \to X$ is a <u>projection</u> if $P^2 = P$.

If $Y$ and $Y'$ are complementary closed subspaces in $X$, $X = Y \oplus Y'$, then the maps taking $x = u + v$ to $u$ and $v$ give rise to two bounded <u>complementary projections</u>

(59.1) $$x \longmapsto u \equiv Px \quad , \quad x \to v \equiv Qx$$

(59.2) $$P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0 \quad , \quad P + Q = I$$

The fact that $P$ and $Q$ are bounded follows from the open mapping theorem as the map

$$\langle y, y' \rangle \longmapsto y + y'$$

is a continuous bijection from the Banach space

$$Y \times Y' = \{\langle y, y' \rangle : y \in Y,\ y' \in Y'\}$$

$$\|\langle y, y' \rangle\| = \|y\| + \|y'\|$$

onto $X$. Conversely, if $P$ and $Q$ are bounded complementary projections in $X$ as in (59.2), then

(60.1) $$Y = \operatorname{ran} P \quad \text{and } Y' = \operatorname{ran} Q$$

are complementary subspaces in $X$ (Exercise).

If $Y$ is a linear subspace of $X$ then the codimension of $Y$ is defined as

(60.2) $$\operatorname{codim}(Y) = \dim(X/Y)$$

Proposition 60.3

Suppose $Y$ is a closed subspace of a Banach space $X$. Then if $\dim Y < \infty$ or $\operatorname{codim} Y < \infty$, then $Y$ can be complemented

Proof: Suppose $\dim Y < \infty$ and let $x_1, \dots, x_n$ be a basis for $Y$, $n = \dim Y$. For $i = 1, \dots, n$, set

$$l_i\left(\sum_{i=1}^{n} \alpha_i x_i\right) = \alpha_i$$

Each $l_i$ is bounded on $Y$ (why?), and hence can be extended to a bounded linear functional $L_i$ on $X$, $i = 1, \dots, n$. Let $Y' = \{x \in X : L_i(x) = 0 \text{ for each } i = 1, \dots, n\}$.

Now $Y'$ complements $Y$. Indeed $Y'$ is clearly closed and if $x \in X$, set $\tilde{x} = x - \sum_{i=1}^{n} L_i(x) x_i$. Then

$$L_i(\tilde{x}) = L_i(x) - L_i(x) = 0$$

so $\tilde{x} \in Y'$: thus $x = \tilde{x} + \sum_{i=1}^{n} L_i(x) x_i \in Y \oplus Y'$.

Finally if $x \in Y \cap Y'$ then $x = \sum_{i=1}^{n} \alpha_i x_i$ and $L_i(x) = \alpha_i = 0$ $\therefore x = 0$, so that the sum is direct.

Now suppose $\operatorname{codim} Y = \dim X/Y = n < \infty$. Let $[x_i] = x_i + Y$, $i = 1, \dots, n$, be a basis for $X/Y$. Then if $x \in X$, there $\exists$ $\alpha_1, \dots, \alpha_n$ such that $[x] = \alpha_1[x_1] + \dots + \alpha_n[x_n]$.

and so $x - \alpha_1 x_1 - \cdots - \alpha_n x_n \in Y$. It follows that any $x \in X$ can be written as $u + v$ where $u \in Y$ and $v \in Y' = \text{span}\{x_1, \ldots, x_n\} = \langle x_1, \ldots, x_n \rangle$. If $x \in Y \cap Y'$, then $x = \alpha_1 x_1 + \cdots + \alpha_n x_n$ for some $\{\alpha_i\}$ in $\mathbb{C}$ and so $\alpha_1 [x_1] + \cdots + \alpha_n [x_n] = 0$ which $\Rightarrow \alpha_1 = \cdots = \alpha_n = 0$ and hence $x = 0$. As $Y'$ is automatically closed, it follows that $Y'$ complements $Y$. $\square$

Let $T \in \mathcal{L}(X)$. We say that a scalar $\lambda \in \mathbb{C}$ lies in the <u>spectrum of $T$</u>, denoted $\sigma(T)$, if $T - \lambda = T - \lambda I$ is <u>not</u> a bijection from $X$ onto $X$. The complement $\mathbb{C} \setminus \sigma(T)$ of $\sigma(T)$ is called the <u>resolvent set</u> of $T$ and is denoted by $\rho(T)$. Thus for each $\lambda \in \rho(T)$, $T - \lambda$ is a bijection with inverse $\frac{1}{T-\lambda} = (T-\lambda)^{-1}$ which is necessarily bounded, by the open mapping theorem. For $\lambda, \lambda' \in \rho(T)$ one has the resolvent identity

(63.1) $$\frac{1}{T-\lambda} - \frac{1}{T-\lambda'} = (\lambda-\lambda')\frac{1}{T-\lambda}\frac{1}{T-\lambda'} = (\lambda-\lambda')\frac{1}{T-\lambda'}\frac{1}{T-\lambda}$$

In particular $\frac{1}{T-\lambda}$ and $\frac{1}{T-\lambda'}$ commute. If $\ker(T-\lambda) \neq \{0\}$ then $\lambda$ is an <u>eigenvalue</u> of $T$ and any vector $u \neq 0$ in $\ker(T-\lambda)$, $Tu = \lambda u$, is called an <u>eigenvector</u> of $T$.

The dimension of $\ker(T-\lambda)$ is called the <u>geometric multiplicity</u> of $\lambda$. Recall that for a matrix $M$, if $\det(M-z) = (\lambda-z)^p q(z)$, $q(\lambda) \neq 0$, $p \geq 1$, then $\lambda$ is an eigenvalue of $M$ and $p$ is its algebraic multiplicity: in general the algebraic multiplicity of an eigenvalue is greater or equal to its geometric multiplicity. The algebraic multiplicity of eigenvalues of certain operators will be considered later.

<u>Theorem 64.1</u> Let $T \in \mathcal{L}(X)$. Then

(1) $\rho(T)$ is an open set and $\sigma(T)$ is a closed set

(2) The map $\lambda \mapsto (T-\lambda)^{-1}$ is analytic in $\rho(T)$

(3) $\sigma(T)$ is non-empty

(4) $\sigma(T) = \sigma(T')$ and $\rho(T) = \rho(T')$

<u>Remark 64.5</u> Property (4) generalizes the fact that if $M$ is a square matrix, $\det(M - z) = \det(M' - z)$.

<u>Proof of Th$^{m}$ 64.1</u> ~~If $\sigma(T) = \phi$ then $T-\lambda$ is invertible for all $\lambda \in \mathbb{C}$. Hence, for any $x' \in X'$, $x \in X$, the function~~

If $\lambda \in \rho(T)$, then for $|\lambda' - \lambda|$ sufficiently small

$$\left(1 - (\lambda' - \lambda)(T - \lambda)^{-1}\right)^{-1}$$

exists, by the Neuman series, $(1 - V)^{-1} = \sum_{k=0}^{\infty} V^k$ for $\|V\| < 1$.

~~exists~~ Set $R_{\lambda,\lambda'} = \left(1 - (\lambda' - \lambda)(T - \lambda)^{-1}\right)^{-1} (T - \lambda)^{-1} \in \mathcal{L}(X)$

Then $(T - \lambda') R_{\lambda,\lambda'} = (T - \lambda')(T - \lambda)^{-1} \left(1 - (\lambda' - \lambda)(T - \lambda)^{-1}\right)^{-1}$

$$= \left(1 - (\lambda' - \lambda)(T - \lambda)^{-1}\right)\left(1 - (\lambda' - \lambda)(T - \lambda)^{-1}\right)^{-1} = 1$$

Similarly $R_{\lambda,\lambda'}(T-\lambda') = I$. Hence $\lambda' \in \rho(T)$. Thus $\rho(T)$ is open, and hence $\sigma(T)$ is closed. If $\lambda \in \rho(T)$ and $|\lambda'-\lambda|$ is small then again by the Neumann series,

$$(T-\lambda')^{-1} = R_{\lambda,\lambda'} = (1-(\lambda'-\lambda)(T-\lambda)^{-1})^{-1}(T-\lambda)^{-1}$$
$$= \sum_{k=0}^{\infty} (T-\lambda)^{-k-1}(\lambda'-\lambda)^k.$$

from which we conclude that $\lambda \mapsto (T-\lambda)^{-1}$ is analytic.

If $\sigma(T) = \phi$, then $T-\lambda$ is invertible, and hence analytic, for all $\lambda \in \mathbb{C}$. Hence for any $x' \in X'$, $x \in X$, the function

$$f_{x,x'}(\lambda) = \langle x', \frac{1}{T-\lambda} x \rangle$$

is entire. However for $|\lambda| > \|T\|$, one can expand $(T-\lambda)^{-1}$ in a Neumann series

$$(T-\lambda)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}$$

and so $\|(T-\lambda)^{-1}\| \le \frac{1}{|\lambda|} \frac{1}{1-\frac{\|T\|}{|\lambda|}} \to 0$ as $|\lambda| \to \infty$.

Hence $f_{x',x}(\lambda) \to 0$ as $\lambda \to 0$, and by Liouville's Theorem, $f_{x',x}(\lambda) = \langle x', (T-\lambda)^{-1}x \rangle = 0$. Thus, ~~[illegible]~~ by (24.1), we must have $(T-\lambda)^{-1}x = 0$ $\forall x$ and so $(T-\lambda)^{-1} = 0$. But $(T-\lambda)(T-\lambda)^{-1} = 1$, and so this is a contradiction. Thus $\sigma(T) \neq \phi$, which proves (3).

Finally (4) follows from the Theorem 41.2, $(T-\lambda)$ is a bijection $\Leftrightarrow$ $T'-\lambda = (T-\lambda)'$ is a bijection. $\square$.

We now begin the study of <u>compact operators</u>. Let $T \in \mathcal{L}(X, Y)$ for a pair of Banach spaces $X$ and $Y$. Then $T$ is <u>compact</u> if it takes bounded sets to pre-compact (or, <u>relatively compact</u>) sets. Thus if $\{x_n\}$ is bounded in $X$, $\|x_n\| \leq c$, then $\{Tx_n\}$ has a convergent subsequence. The basic theory of compact operators is due to F. Riesz and T. Schauder and is known as <u>Riesz-Schauder</u> theory. We denote the space of compact operators $T \in \mathcal{L}(X,Y)$ by $K(X,Y)$ and $K(X)$ if $X = Y$.

The following result is immediate.

<u>Theorem 67.1</u> Let $W, X, Y, Z$ be Banach spaces. Then

(1) $S, T \in K(X,Y) \to \mu S + \lambda T \in K(X,Y) \quad \forall \mu, \lambda \in \mathbb{C}$

(2) $C \in \mathcal{L}(W,X)$, $B \in K(X,Y)$, $A \in \mathcal{L}(Y,Z)$

$\Rightarrow \quad BC \in K(W,Y)$ and $AB \in K(X,Z)$

(3) $K(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$. Thus if

$T_n \in K(X,Y)$, $T \in \mathcal{L}(X,Y)$ and $\|T_n - T\| \to 0$ as $n \to \infty$,

then $T \in K(X,Y)$

<u>Proof</u>: Exercise $\square$

Theorem 67.1 implies, in particular, that $K(X)$ is a closed ideal in $\mathcal{L}(X)$.

An operator $T \in \mathcal{L}(W,X)$ is <u>finite rank</u> if

$\dim \operatorname{ran} T < \infty$. Such operators can be represented in the form (exercise)

(68.1) $$Tx = \sum_{i=1}^{n} x_i'(x)\, y_i \quad , x \in X$$

for some independent set of vector $y_i \in Y$, $i=1,\dots,n$, $n = \dim(\operatorname{ran} T)$, and some independent set of bounded linear functionals $x_i' \in X'$, $i=1,\dots,n$. Finite rank operators are clearly compact (why?). In Hilbert space, every compact operator $T \in K(H)$ is the norm limit of finite rank operators, $\|T_n - T\| \to 0$ as $n \to \infty$, $T_n$ finite rank. (see Reed-Simon, e.g): This is not true in general in Banach space.

Lecture 6

Theorem 68.2 (Schauder)
An operator $T \in \mathcal{L}(X,Y)$ is compact if and only if $T' \in \mathcal{L}(Y', X')$ is compact.

Proof: Recall the Arzela-Ascoli Theorem (see [Yosida]): Let $S$ be a compact metric space and let $C(S)$ denote the B-space of continuous functions on $S$ with norm $\|x\| = \sup_{s \in S} |x(s)|$.