= \text{ran} T. \quad \text{Let } y' \text{ be an extension of } l \text{ to } Y.

Then for all } x \in X, \quad \langle T' y', x \rangle = \langle y', Tx \rangle

= \mathcal{L}(Tx) = \langle x', x \rangle, \quad \text{and so } T'y' = x' \text{ if } x' \in \text{ran} T'.

This completes the proof of Banach's Theorem (45.3).

\underline{Lecture 5} \quad \underline{Remark 57.1:} \quad \text{Suppose } T \in \mathcal{L}(X, X'), \text{ and hence } T' \in \mathcal{L}(Y, X'),

\text{has closed range.} \quad \text{It is interesting to apply (46.1) to } T'. \quad \text{We have}

\begin{equation}
\text{ran } T' = \{ x' \in X' : \langle x'', x' \rangle = 0 \quad \forall x'' \in \ker T' \}
\end{equation}

But (46.2) also applies, so we also have

\begin{equation}
\text{ran } T' = \{ x' \in X' : \langle x', x \rangle = 0 \quad \forall x \in \ker T' \}
\end{equation}

Using the injection \( \varphi : X \to X'' \) mapping \( x \mapsto x'' = \varphi(x) \)

\( x''(x') = x'(x) \) (see (26.31)), we see that

\[ \langle x'', x' \rangle = 0 \]

for all \( x'' \in \ker T'' \), \( y' \) and only if

\[ \langle x'', x' \rangle = 0 \]

for all \( x'' \in \ker T'' \cap \ker \varphi \). But in general \( T' \notin \ker T'' \).
and so the conclusion is somewhat surprising.

Let $Y$ be a closed linear subspace of a Banach space $X$. We say that closed linear subspace $Y'$ of $X$ is a complement of $Y$ in $X$ if $X$ is a direct sum of $Y$ and $Y'$, i.e., any $x \in X$ can be expressed as a unique sum.

\[(58.1) \quad x = u + v \quad \text{where } u \in Y \text{ and } v \in Y'.\]

Note that uniqueness is equivalent to $Y \cap Y' = \{0\}$.

Complements $Y'$ of $Y$, if they exist, are not unique.

For example, if $X = \mathbb{R}^2$ and $Y = \{(x, y) : x \in \mathbb{R}\}$, then

$Y' = \{(x, 0) : x \in \mathbb{R}\}$ and $Y'' = \{(0, y) : y \in \mathbb{R}\}$

are both complements of $Y$ in $X$. (See also Instr. Set 11.)

By Theorem 13.2, every closed subspace $Y$ of a Hilbert space $H$ has a complement, viz. $Y' = Y^\perp$. But in general a Banach space may have subspaces that
cannot be complemented. For example in [R. J. Phillips, On
linear transformations, Transactions of Amer. Math. Society, 48 (1950), 516-554], Phillips showed that \( c_0 = \{ x = \{ x_n \} : x_n \to 0 \} \) is a subspace in \( L^\infty \) that cannot be
complemented (see Problem set #4).

A linear operator \( P : X \to X \) is a projection if \( P^2 = P \).

If \( Y \) and \( Y' \) are complementary closed subspaces in \( X \),
\( X = Y \oplus Y' \). Then the maps taking \( x = u + v \) to \( u \) and
\( v \) give rise to two bounded complementary projections.

\[(59.1) \quad x \mapsto u = Px, \quad x \mapsto v = Qx\]

\[(59.2) \quad P^2 = P, \quad Q^2 = 0, \quad PQ = QP = 0, \quad P + Q = I\]

The fact that \( P \) and \( Q \) are bounded follows from
the open mapping theorem as the map

\[ \langle y, y' \rangle \mapsto y + y' \]

is a continuous bijection from the Banach space
\[ Y \times Y' = \{ (y, y') : y \in Y, y' \in Y' \} \]

\[ \| (y, y') \| = \| y \| + \| y' \| \]

onto \( X \). Conversely, if \( P \) and \( Q \) are bounded complementary projections in \( X \) as in (59.2), then

\[ (60.1) \quad Y = \text{ran} P \quad \text{and} \quad Y' = \text{ran} Q \]

are complementary subspaces in \( X \) (Exercise).

If \( Y \) is a linear subspace of \( X \) then the codimension of \( Y \) is defined as

\[ (60.2) \quad \text{codim}(Y) = \dim(X/Y) \]

**Proposition 60.3**

Suppose \( Y \) is a closed subspace of a Banach space \( X \).

Then if \( \dim Y < \infty \) or \( \text{codim} Y < \infty \), then \( Y \) can be complemented.

**Proof:** Suppose \( \dim Y < \infty \) and let \( x_1, \ldots, x_n \) be a
Let $Y$, $n = \dim Y$. For $i = 1, \ldots, n$, set

$$l_i \left( \sum_{i=1}^{n} d_i x_i \right) = d_i$$

Each $l_i$ is bounded on $Y$ (why?), and hence can be extended to a bounded linear functional $L_i$ on $X$, $i = 1, \ldots, n$. Let $Y' = \{ x \in X : L_i(x) = 0 \text{ for each } i = 1, \ldots, n \}$.

Now $Y'$ complements $Y$. Indeed $Y'$ is clearly closed and if $x \in X$, set $\tilde{x} = x - \sum_{i=1}^{n} L_i(x) x_i$. Then

$$L_i(\tilde{x}) = L_i(x) - L_i(x) = 0$$

for $\tilde{x} \in Y'$. Thus $x = \tilde{x} + \sum_{i=1}^{n} L_i(x) x_i \in Y \oplus Y'$.

Finally if $x \in Y \cap Y'$, then $x = \sum_{i=1}^{n} d_i x_i$ and

$$L_i(x) = d_i = 0 \Rightarrow x = 0,$$ no That the sum is direct.

Now suppose $\codim Y = \dim X / Y = n < \infty$. Let

$$x_i = x_i + y_i, \ i = 1, \ldots, n,$$ be a basis for $X / Y$. Then

$y \in X$, there $\exists d_1, \ldots, d_n$ such that $x = d_1 [x_1] + \cdots + d_n [x_n]$.\]
and so \( x = x_1 x_1 + \ldots + x_n x_n \in Y \). It follows that any \( x \in X \) can be written as \( u + v \) where \( u \in Y \) and \( v \in Y' = \text{span} \{ x_1, \ldots, x_n \} = \langle x_1, \ldots, x_n \rangle \).

If \( x \in Y \cap Y' \), then \( x = d_1 x_1 + \ldots + d_n x_n \) for some \( d_1, \ldots, d_n \in C \) and no \( x_1^* x_1 + \ldots + x_n^* x_n = 0 \) which entails \( d_1 = \ldots = d_n = 0 \) and hence \( x = 0 \). As \( Y' \) is automatically closed, it follows that \( Y' \) complements \( Y \). \( \square \)

Let \( T \in \mathcal{L}(X) \). We say that a scalar \( \lambda \in C \) lies in the spectrum of \( T \), denoted \( \sigma(T) \), if \( T - \lambda = T - \lambda I \) is not a bijection from \( X \) onto \( X \). The complement \( \Delta \setminus \sigma(T) \) of \( \sigma(T) \) is called the resolvent set of \( T \) and is denoted by \( \rho(T) \). Thus for each \( \lambda \in \rho(T) \), \( T - \lambda \) is a bijection with inverse \( \frac{1}{T - \lambda} = (T - \lambda)^{-1} \) which is necessarily bounded, by the open mapping theorem. For \( \lambda, \lambda' \in \rho(T) \) one has
The resolvent identity

\[ (63.1) \quad \frac{1}{T - \lambda} = \frac{1}{T - \lambda'} = (\lambda - \lambda') \cdot \frac{1}{T - \lambda} = (\lambda - \lambda') \cdot \frac{1}{T - \lambda} \]

In particular, \( \frac{1}{T - \lambda} \) and \( \frac{1}{T - \lambda'} \) commute. If \( \text{ker}(T - \lambda) \neq \{0\} \)

Then \( \lambda \) is an eigenvalue of \( T \) and any vector \( u \neq 0 \) in \( \text{ker}(T - \lambda) \), \( Tu = \lambda u \), is called an eigenvector of \( T \).

The dimension of \( \text{ker}(T - \lambda) \) is called the geometric multiplicity of \( \lambda \). Recall that for a matrix \( M \),

\[ \det(M - \lambda I) = (\lambda - 3)^p \phi(3), \quad \phi(3) \neq 0, \quad p \geq 1, \]

then \( \lambda \) is an eigenvalue of \( M \) and \( p \) is its algebraic multiplicity.

In general, the algebraic multiplicity of an eigenvalue is greater or equal to its geometric multiplicity. The algebraic multiplicity of eigenvalues of \( T \) operators will be considered later.
Theorem 64.1. Let $T \in \mathcal{L}(X)$. Then

1. $\rho(T)$ is an open set and $\sigma(T)$ is a closed set.
2. The map $\lambda \mapsto (T-\lambda)^{-1}$ is analytic in $\rho(T)$.
3. $\sigma(T)$ is non-empty.
4. $\sigma(T) = \sigma(T')$ and $\rho(T) = \rho(T')$.

Remark 64.5. Property (4) generalizes the fact that if $A$ is a square matrix, $\det(\lambda I - A) = \det(\lambda' I - A')$.

Proof of Theorem 64.1. If $\sigma(T) = \emptyset$, then $T$ is invertible for all $\lambda \in \mathbb{C}$. Hence, for any $x \in X'$, $x \in X$, the function

$$(1 - (\lambda - \lambda') (T-\lambda)^{-1})^{-1}$$

exists, by the Neumann series, $(1 - V)^{-1} = \sum_{k=0}^{\infty} V^k$ for $\|V\| < 1$.

Set $R_{\lambda, \lambda'} = (1 - (\lambda - \lambda') (T-\lambda)^{-1})^{-1} (T-\lambda)^{-1} \in \mathcal{L}(X)$.

Then $(T-\lambda) R_{\lambda, \lambda'} = (T-\lambda Y (T-\lambda)')^{-1} (T-\lambda Y (T-\lambda)')^{-1}$

$$= (1 - (\lambda - \lambda') (T-\lambda)^{-1})^{-1} (1 - (\lambda - \lambda') (T-\lambda)^{-1})^{-1} = 1$$
Similarly \( R_{\lambda', \lambda} (T - \lambda') = I \). Hence \( \lambda' \in \rho(T) \). Thus \( \rho(T) \) is open, and hence \( \sigma(T) \) is closed. If \( \lambda \in \rho(T) \) and \( |\lambda' - \lambda| \) is small, then again by the Neumann series,

\[
(T - \lambda')^{-1} = R_{\lambda, \lambda'} = (I - (\lambda' - \lambda)(T - \lambda')^{-1})^{-1} = \sum_{k=0}^{\infty} (T - \lambda)^{-k} (\lambda' - \lambda)^k
\]

from which we conclude that \( \lambda \mapsto (T - \lambda)^{-1} \) is analytic.

If \( \sigma(T) = \emptyset \), then \( T - \lambda \) is invertible, and hence analytic, for all \( \lambda \in \mathbb{C} \). Hence for any \( x' \in X, x \in X \), the function

\[
\Phi_{x, x'}(\lambda) = \langle x', \frac{1}{T - \lambda} x \rangle
\]

is entire. However, for \( |\lambda| > \|T\| \), one can expand \( (T - \lambda)^{-1} \) in a Neumann series

\[
(T - \lambda)^{-1} = -\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{T^n}{\lambda^n}
\]

and so

\[
\| (T - \lambda)^{-1} \| \leq \frac{1}{\| T \|} \frac{1}{|\lambda|} \to 0 \quad \text{as} \quad |\lambda| \to \infty.
\]
Hence \( f_{x,x}(x) \to 0 \) as \( x \to 0 \), and by Liouville's Theorem, \( f_{x,x}(x) = \langle x', (T-\lambda)^{-1}x' \rangle = 0 \). Thus, by (24.1), we must have \((T-\lambda)^{-1}x = 0\) for all \( x \) and \( \lambda \).

Finally (4) follows from the Theorem 41.2, \((T-\lambda)\) is a bijection \( \iff T-\lambda = (T-\lambda)^{-1} \), 

We now begin the study of compact operators. Let 

\( T \in \mathcal{L}(X,Y) \) for a pair of Banach spaces \( X \) and \( Y \).

Then \( T \) is compact if it takes bounded sets to pre-

compact sets. Thus if \( \{x_n\} \) is bounded in \( X \), \( \|x_n\| \leq c \),

then \( \{T x_n\} \) has a convergent subsequence. The basic theory of compact operators is due to F. Riesz and T. Schauder and is known as Riesz-Schauder theory. We denote the
space of compact operators $T \in \mathcal{L}(X,Y)$ by $K(X,Y)$ and $K(X)$ if $X = Y$.

The following result is immediate.

Theorem 67.1: Let $W, X, Y, Z$ be Banach spaces. Then

1. $s, T \in K(X,Y) \implies \lambda s + \alpha T \in K(X,Y)$ for $\lambda, \alpha \in \mathbb{C}$

2. $C \in \mathcal{L}(W,X)$, $B \in K(X,Y)$, $A \in \mathcal{L}(Y,Z)$

   $BC \in K(W,Y)$ and $ABC \in K(X,Z)$

3. $K(X,Y)$ is a closed subset of $\mathcal{L}(X,Y)$. Thus if $T_n \in K(X,Y)$, $T \in \mathcal{L}(X,Y)$ and $\|T_n - T\| \to 0$ as $n \to \infty$,

   Then $T \in K(X,Y)$

Proof: Exercise $\Box$

Theorem 67.1 implies, in particular, that $K(X)$ is a closed ideal in $\mathcal{L}(X)$. An operator $T \in \mathcal{L}(W,X)$ is finite rank if
dim ran $T < \infty$. Such operators can be represented in the form (exercise)

\[(68.1) \quad Tx = \sum_{i=1}^{n} x_i'(x)y_i, \quad x \in X\]

for some independent set of vectors $y_i \in Y$, $i = 1, \ldots, n$,

$n = \dim (\text{ran } T)$, and some independent set of bounded linear functionals $x_i' \in X'$, $i = 1, \ldots, n$. Finite rank operators are clearly compact (why?). In Hilbert space, every compact operator $T \in \mathcal{K}(H)$ is the norm limit of finite rank operators, $\| T_n - T \| \to 0$ as $n \to \infty$, $T_n$ finite rank. (See Reed-Simon, e.g.) This is not true in general in Banach space.

**Theorem 68.2** (Schauder)

An operator $T \in \mathcal{L}(X, X)$ is compact if and only if $T' \in \mathcal{L}(Y, Y')$ is compact.

Proof: Recall the Arzela-Ascoli Theorem: Let $S$ be a compact metric space and let $C(S)$ denote its B-space of continuous functions on $S$ with norm $\| x \| = \sup_{s \in S} |x(s)|$.

\[\text{(see [Yosida].)}\]