On the other hand, by (24.31),

$$\|Tx\| = \sup_{\|y\| = 1} |y'(Tx)| = \sup_{\|y\| = 1} |y'(T'y)|$$

$$\leq \sup_{\|y\| = 1} \|T'y\| \|x\|$$

$$\leq \sup_{\|y\| = 1} \|T'y\| \|x\|$$

and so \(\|T\| \leq \|T'y\| \|x\|\). Thus we have equality in (40.2)

$$(41.1) \quad \|T'y\| = \|T\|$$

Moreover the following is true:

Lecture 4

Theorem 41.2

Let \(T \in L(X, Y)\). Then \(T\) is a bijection from \(X \rightarrow Y\) if and only if \(T'\) is a bijection from \(Y' \rightarrow X'\) and if \(T\), or equivalently \(T'\), is a bijection, then \((T')^{-1} = (T^{-1})'\) and

as \(T' \in L(Y, X)\), \((T')^{-1} \in L(X, Y)\) (for any square matrix \(A\)).

Remark: In finite dimensions, this result is reflected in the fact that \(det(A) = det(A')\).

Proof: Indeed if \(T\) is a bijection, then unravelling the definition we see immediately that

$$(41.3) \quad \langle (T^{-1})'x', y' \rangle = \langle x', T'y' \rangle \quad \forall x' \in X', y' \in Y$$
\[ \langle (T^{-1})'x', Tx \rangle = \langle x', T^{-1}Tx \rangle = x'(x), \quad \forall x \in X \]

\[ = \langle T'(T^{-1}'x'), x \rangle = \langle x', x \rangle \quad \forall x \]

\[ = \langle T'(T^{-1}'x'), x \rangle = \langle x', x \rangle \quad \forall x \]

\[ \Rightarrow \quad T'(T^{-1}'x') = x' \]

And also for \( x' = T'y' \), \( (4.1.3) = \)

\[ \langle (T^{-1})'T'y', y' \rangle = \langle T'y', T^{-1}y' \rangle \]

\[ = \langle y', T(T^{-1}y') \rangle = \quad \forall y \in Y \]

\[ = \langle y', y \rangle \quad \forall y \in Y \]

\[ \Rightarrow \quad (T^{-1})'T'y' = y' \]

Hence \( T' \) is a bijection and \( (T')^{-1} = (T^{-1})' \). Conversely

Suppose that \( T' \) is a bijection. By (4.1.1), given any \( 0 \neq x \in X \),

choose \( x' \in X' \) s.t. \( x'(x) = (1x) \) and \( Mx' = 1 \). As \( T' \) is a

bijection, \( \exists y' \in Y' \) s.t. \( x' = T'y' \). Hence

\[ \| x' \| = \langle x'(x) = \langle x', x \rangle = \langle T'y', x \rangle = \langle y', T'x \rangle \]

and so \( Tx = 0 \Rightarrow x = 0 \). We conclude that \( T \) is injective.

On the other hand, (42.1) shows that for any \( x \),

\[ \| x' \| = \| y' \| \| T'x' \| \]

but by (28.2), for some \( c > 0 \), \( \| y' \| \leq c \| T'y' \| = c \| x' \| = \frac{1}{2} \)
Thus

\[ \| Tx \| = \frac{1}{2} \| T \| \| x \| \]

and we conclude, again, by (28.2), that \( \text{ran} \, T \) is closed.

Suppose \( \text{ran} \, T = \text{ran} \, T \not\subseteq Y \) and let \( 0 \neq y_0 \in Y \setminus \text{ran} \, T \). Then

by Proposition 26.1 there exists \( y' \in Y' \) such that \( y'(y_0) = \| y_0 \| \)

and \( y'(Tx) = 0 \) \( \forall x \in X \). But then for all \( x \in X \),

\[ \langle Ty', x \rangle = \langle y', Tx \rangle = 0 \]

and no \( Ty' = 0 \). But \( y'(y_0) = \| y_0 \| \neq 0 \) and this contradicts the fact that \( \ker T' = \{ 0 \} \). Thus \( T \) is a bijection and

\[ (T^{-1})' = (T')^{-1}. \]

**Exercise:** If \( T \in \mathcal{L}(X, Y) \) and \( S \in \mathcal{L}(Y, Z) \), then \( (ST)' \in \mathcal{L}(Z', X') \) and \( (ST)' = T' S' \).

Let \( (H, \langle \cdot, \cdot \rangle) \) be a Hilbert space and let \( T \in \mathcal{L}(H) \). Then for any \( x \in H \), the map \( y \mapsto \langle x, Ty \rangle \) defines a bounded linear functional on \( H \) and hence by the Riesz representation theorem, there exists a unique \( x^* \in H' \) such that

\[ \langle x, Ty \rangle = \langle x^*, y \rangle \quad \forall y \in H \]
Set
\[(4.4.1)\]
\[T^* x = x^*\]

Clearly, \(T^* \in \mathcal{L}(\mathcal{V})\) and
\[(4.4.2)\]
\[\|T^*\| = \|T\| \|

The operator \(T^*\) is called the adjoint of \(T\) and is related to \(T^\prime\) in the following way. We have
\[(4.4.3)\]
\[(x, Ty) = (T^* x, y), \quad x, y \in \mathbb{V}\]

In terms of the (anti-linear) map \(\xi\) on page 27 taking \(x \to \xi x\), \(x, y \in \mathbb{V}\),
\[(4.4.4)\]
\[<4(4x), y> = (T^* x, y)\]
\[= (x, Ty) = 4(x, T(y)) = <T^\prime y, 4(x), y> \text{ and so}\]
\[4 \cdot T^* = T^\prime \cdot 4\]

or
\[(4.4.4)\]
\[T^* = 4^{-1} T^\prime \cdot 4\]

An operator \(T \in \mathcal{L}(\mathcal{V})\) is self-adjoint if \(T = T^*\).

The following important result of Banach provides information on the geometry of \(\text{ran} T\) for any bounded linear map \(T\) from a Banach space \(X\) to a Banach space \(Y\), by defining
relational
\[ \langle \gamma', x \rangle = \langle y', Tx \rangle, \quad x \in X, \quad y' \in Y' \]

shows that

(45.1) \( \ker T' = \{ y' : \langle y', y \rangle = 0 \quad \forall y \in \operatorname{ran} T \} \)

On the other hand

(45.2) \( \operatorname{ran} T \subseteq \{ y' : \langle y', y \rangle = 0 \quad \forall y \in \mathcal{N}(T') \} \)

but in general the inclusion is strict. Indeed the set on the right-hand side of (45.2) is always closed, whereas \( \operatorname{ran} T \) may not be closed. However, as we now show, if \( \operatorname{ran} T \) is closed, then we have equality in (45.2).

Theorem 45.3 (Banach) Let \( X \) and \( Y \) be Banach spaces and let \( T \in L(X, Y) \). Then the following properties are equivalent:

(45.4) \( \operatorname{ran} T \) is closed in \( Y \)

(45.5) \( \operatorname{ran} T' \) is closed in \( X' \)
\[(46.1) \quad \text{ran } T = \{ y \in Y : \langle y', y \rangle = 0 \neq y' \in \ker T' \} \]

\[(46.2) \quad \text{ran } T' = \{ x' \in X' : \langle x', x \rangle = 0 \neq x' \in \ker T' \} \]

In order to prove the theorem, it is convenient to use the following lemma.

**Lemma 46.3** Let \( X_i, 1 \leq i \leq 4 \), be Banach spaces.

Suppose that

- \( T_1 \in \mathcal{L}(X_1, X_2) \) is surjective
- \( T_2 \in \mathcal{L}(X_2, X_3) \)
- \( T_3 \in \mathcal{L}(X_3, X_4) \) is injective with closed range
- \( T_4 \in \mathcal{L}(X_1, X_4) \)

Suppose that \( T_4 = T_3 T_2 T_1 \). Then

\[(46.4) \quad \text{ran } T_4 \text{ is closed } \iff \text{ran } T_2 \text{ is closed.} \]

**Proof:**

\[ X_1 \xrightarrow{T_1} X_2 \xrightarrow{T_2} X_3 \xrightarrow{T_3} X_4 \]

Suppose \( \text{ran } T_2 \) is closed and let \( T_4 x_n \to y, \ x_n \in X_1, \) \( y \in X_4 \). By (28.2), \( \| T_3 T_2 T_1 x \| \geq c \| T_2 T_1 x \| \), \( c > 0, \) \( T_2 T_1 x \in X_2 \), and so
Suppose \( T_3 T_1 x_n \to u \in X_3 \). But as \( \text{ran} \ T_2 \) is closed, we must have \( u = T_2 v \) for some \( v \in X_2 \).

However, \( T_1 \) is surjective and hence \( v = T_1 w \) for some \( w \in X_1 \),

\[
\lim_{n \to 0} y = \lim_{n \to 0} T_4 x_n = \lim_{n \to 0} T_3 (T_2 T_1 x_n) = T_3 u = T_3 T_2 v = T_3 T_2 T_1 w = T_4 w \quad \text{and no ran} \ T_4 \text{ is closed.}
\]

Conversely, suppose that \( \text{ran} \ T_4 \) is closed and let \( T_0 x_n \to u, \)

\( u_n \in X_2, u \in X_3 \). As \( T_1 \) is surjective, this implies \( T_2 T_1 w_n \to u \) for some \( w_n \in X_1 \), and hence \( T_2 w_n \to T_3 u \). But as \( \text{ran} \ T_4 \) is closed, we must have \( T_3 u = T_4 w = T_3 T_2 T_1 w \) for some \( w \in X_1 \), and hence \( u = T_2 T_1 w \) as \( T_3 \) is injective. It follows that \( u \in \text{ran} \ T_2 \) and so \( \text{ran} \ T_2 \) is closed.

**Proof of Theorem 4.6.1** We show first that to prove the equivalence of (4.5.4) and (4.5.5), it is enough to consider the case where
where \( T \) is injective. Let \( \text{ran} T \) be dense in \( X \). Suppose \( T \in \mathcal{L}(X, Y) \). Then

\[(4.8.1) \quad T = I \circ [T] \pi \]

where

- \( \pi \) is the map \( x \mapsto [x] \) taking \( X \) onto \( X/\ker T \)
- \( [T] \) is the map \( [x] \mapsto Tx \) taking \( X/\ker T \) into \( \overline{\text{ran}[T]} = \overline{\text{ran} T} \), and
- \( I \) is the map \( y \mapsto y \) injecting \( \overline{\text{ran} T} \) into \( Y \)

Clearly, \( \text{ran} [T] \) is closed \( \iff \) \( \text{ran} T \) is closed. Taking adjoints in (4.8.1) we obtain

\[(4.8.2) \quad T' = \pi'[T'] I' \in \mathcal{L}(Y', X') \]

where

- \( I' \in \mathcal{L}(Y', \overline{\text{ran} T}') \)
- \( [T'] \in \mathcal{L}(\overline{\text{ran} T}', X'/\ker T') \)
- \( \pi' \in \mathcal{L}(X'/\ker T', X') \)

For \( y' \in Y' \) and \( y \in \overline{\text{ran} T} \), then \( \langle T'y', y \rangle = \langle y', Ty \rangle = \langle b', y \rangle \)
and we see that $I'y'$ is simply the restriction of $y'$ to $\overline{\text{ran} \, T}$.

But by the Hahn-Banach Theorem, any $y_0 \in (\overline{\text{ran} \, T})'$ can be extended to an element $y' \in Y'$; clearly $I'y' = y_0$. It follows that $I'$ is surjective. On the other hand, if $[(x)'] \\
\in (X/\ker T)'$ and $x \in X$, then $<(\pi'[x]'), x> = <[x]', \pi x> = <[x]', [x]>.$

As $\pi'$ is surjective, it follows immediately that $\pi'$ is injective. We show that ran $\pi'$ is closed. First note that $\left\langle [\overline{\lambda} [x']], x \right\rangle$

if $x' = \pi'[x]'$, then clearly $x'(x) = <\pi'[x]', x> = <[x]', \pi x> = <[x]', [x]> = 0$ for all $x \in \ker T$. Conversely suppose that $x'(x) = 0$ for all $x \in \ker T$. Define the linear functional on $X/\ker T$

$$l([x]) = x'(x)$$

This functional is well defined: if $[x] = [y]$, then $x - y \in \ker T$ and hence $x'(x) = x'(y)$. Also $|l([x])| \leq \|x\| \|x\|$ for any $x \in [x]'$, and as we can always choose $x$ such that $\|x\| = 2 \|x\|$, we see

that $l$ is bounded and hence $l \in (X/\ker T)'$. But $<\pi', x>$
\[ \langle x, \pi' x \rangle = \langle e, [x] \rangle = x'(x) = \langle x', x \rangle, \text{ which implies that } \]

\[ \pi' = x'. \text{ Thus we have shown that} \]

\[ \text{ran } \pi' = \{ x' \in X': x'(x) = 0 \text{ for } x \in \ker T \} \]

and it follows, in particular, that ran \( \pi' \) is closed. Applying Lemma 46.1, we conclude that \( \text{ran } T' \) is closed \( \iff \) ran \( [T'] \) is closed in \((X/\ker T)'\).

The above calculations show that to prove the equivalence of (45.4) and (45.5), it is enough to show that

\[ \text{ran } [T] \text{ is closed } \iff \text{ran } [T'] \text{ is closed} \]

As \([T]\) is injective, we see that it is enough to prove the equivalence of (45.4) and (45.5) when \( T \) is injective.

So suppose that \( T \in \mathcal{L}(X,Y) \) is injective with closed range \( \text{ran } T = Y_1 \subset Y \). Now regard \( T \) as a bijection \( T_1 \) from \( X \) onto the Banach space \( Y_1 \). It
follows from Theorem 41.2. That $T_i'$ is a bijection from $Y_i'$ onto $X'$. In particular for any $x' \in X'$, there exists $y_i' \in Y_i'$ such that $T_i'y_i' = x'$. Thus for any $x \in X$,

$$\langle T_i'y_i', x \rangle = \langle y_i', T_i x \rangle = \langle y_i', Tx \rangle = \langle x', x \rangle$$

As $Y_i \subset C^*$, we may extend $y_i'$ to a bounded linear functional $y_i'$ on $Y$. Then $T'y_i' \in X'$ and we have for any $x \in X$,

$$\langle T'y_i', x \rangle = \langle y_i', Tx \rangle = \langle y_i', Tx \rangle = \langle x', x \rangle$$

so $T'y_i' = x'$. Thus $\text{ran } T' = X$ so that, in particular, $\text{ran } T'$ is closed.

(If $T$ is one-to-one and)

Conversely, suppose that $\text{ran } T'$ is closed in $X'$. We must show that $\text{ran } T$ is closed. Let $Y_i = \text{ran } T \cap Y$ and let $T_i : X \to Y_i$ as before. We show that for $T_i' : Y_i' \to X'$,

$\text{ran } T'$ closed $\implies \text{ran } T_i'$ closed.

Let $T_i'y_i, x' \in X'$, $y_i' \in Y_i'$, $x' \in X'$
For each \( n \), let \( y'_n \in Y' \) be an extension to \( y_n' \) to \( Y' \). Then
\[
( T' y'_n, x ) = ( y'_n, T x )
\]
\[
= ( y'_n, T x ) \quad \text{and} \quad T x \in Y',
\]
\[
= ( y'_n, T x ) = ( T' y'_n, x ) \quad \text{. Then}
\]
\[
T' y'_n = T' y'_n \to x'
\]
and as \( \text{ran} \ T' \) is closed, \( T' y' \in Y' \) or \( T' y' = x' \)
\( \text{i.e.} \)
\[
( x', x ) = ( T' y', x ) = ( y', T x ) = ( y', T x ' ) \quad \text{where}
\]
\( y'n \text{ is restriction of } y' \text{ to } y' \),
\[
= ( y', T x ) = ( T', y', x ) \quad \text{and no } x' = T' y'.
\]
This shows that \( \text{ran} \ T' \) is closed.

Following [Yosida], we now show the following fact: for any \( \varepsilon > 0 \),
\[
\text{there exists } n \epsilon 1 \text{ such that}
\]
\[
(52.1) \quad \{ y \in Y : \| y \| < \varepsilon \} \subseteq \{ T x : \| x \| < \varepsilon \} = S_r
\]
Or not, \( T y_n \in Y', y_n \to 0 \), \( y_n \neq S_r \). But \( S_r \) is clearly a
closed, convex balanced set and hence by Thm 27.1,

\[ \{ y_n \in Y_1 \mid + \]

\[(53.1) \quad \frac{1}{3} y_n / y_n > 1 \Rightarrow |y_n(T, x) / \| = |T, y_n(x) / \| \text{ for all } \|x\| < \epsilon \]

It follows that \[ \|T, y_n \| < \frac{1}{3} \|y_n\| \|y_n\| \]. On the other hand, as \( \text{ran } T_1 \) is dense in \( Y_1 \), \( T_1 \) is injective and so as \( \text{ran } T_1 \) is closed, \( \|T_1, y_n \| \geq c \|y_n\| \)

for some \( c > 0 \), \( \forall y_n \in Y_1 \), by (28.2). Thus

\[ \frac{1}{3} \|y_n\| \|y_n\| \geq \|T_1, y_n \| \geq c \|y_n\| \]

\[ \Rightarrow \|y_n\| \geq \frac{c}{3} \]

which contradicts \( y_n \rightarrow 0 \). This proves (52.1).

Next we show that for any \( \epsilon > 0 \) we must have

\[(53.2) \quad \{ y_n \in Y_1 : \|y_n\| < \frac{1}{3} \|y_n\| C \ \{ x \in X : \|x\| \leq \epsilon \} \}

for sufficiently large \( n \). This then proves that
\[ \text{ran } T_i = Y_i, \quad \text{for } y \in Y_i, \quad \text{then } \|y\| = \frac{1}{n} \|y\| \]

and \( \exists y \in T_i x \) for some \( \|x\| \leq \varepsilon \). Hence

\[ y = T_i(n \|y\| x) \in \text{ran } T_i. \quad \text{As } \text{ran } T_i = \text{ran } T, \]

and \( \text{ran } T = Y_i \), which is closed by construction.

To prove (53.2), let \( \varepsilon > 0 \) be given and set

\[ \varepsilon_i = \varepsilon/2^{i+1}, \quad i \geq 0. \]

By (52.1) there is a sequence of positive numbers \( \eta_i \to 0 \) such that

\[ B_i = \{ y \in Y_i : \|y\| \leq \eta_i \} \subset \{ T_i x : \|x\| \leq \varepsilon_i \}. \]

Let \( y \in B_i \). Then \( \|y - T_i x\| \leq \eta_i \) for some \( \|x\| \leq \varepsilon_i \).

\[ \|x\| \leq \varepsilon_0. \quad \text{As } y - T_x \in B_i, \text{ the } \text{then } \|x\| \leq \varepsilon_i. \]

\[ \|y - T_i x - T_i x_0\| \leq \eta_i. \quad \text{Repeating this process, we find a sequence } \{x_i\} \text{ with } \|x_i\| \leq \varepsilon_i \text{ s.t.} \]

\[ \|y - T_i(\sum_{i=0}^{\infty} x_i)\| \leq \eta_{n+1}, \quad n \geq 0. \]

But for \( n > m \), \( \|\sum_{i=m}^{n} x_i\| \leq \sum_{i=m}^{n} \varepsilon_i = \sum_{i=m}^{n} \frac{1}{2^{i+1}} \).
and no \( \exists \sum_{i=0}^{n} x_i \in X \) is a Cauchy sequence. Hence

\[
\sum_{i=0}^{n} x_i \to x \in X \quad \text{and} \quad \|y - Tx\| = 0 \quad \text{if} \quad y = Tx
\]

Moreover \( \|x\| \leq \sum_{i=0}^{\infty} \|x_i\| = \varepsilon \). This shows that

\[
\exists y \in Y : \|y\| \leq M_0 \quad \Rightarrow \quad \|Tx - x\| \leq \varepsilon
\]

as desired. This completes the proof that (45.4) is equivalent to (45.1).

The implications (46.1) \( \Rightarrow \) (45.4) and (46.2) \( \Leftarrow \) (45.5) are clear as the RHS's are closed sets.

The implication (45.4) \( \Rightarrow \) (46.1) is straightforward.

Indeed, by (45.2), \( \text{ran} \, T \) is a subset of the RHS of (46.1). Suppose \( y \in Y \) and \( \langle y, y \rangle = 0 \). Then \( y \in \ker T \), hence \( y \in \ker T \).

Then as \( \text{ran} \, T \) is closed, \( \exists \) by Prop. 26.1, \( y' \in Y' \) such that \( y'(y) = 0 \) and \( y'(Tx) = 0 \) for all \( x \in X \). Thus

\[
\langle T'y', x \rangle = \langle y', Tx \rangle = 0
\]

and we see that \( y' \in \ker T' \).
by \langle y', x \rangle = 0, which is a contradiction.

It remains to prove that (45.5) \implies (46.2).

As ran T' \subset \text{RHS}, we only need to show that if

\langle x', x \rangle = 0 \neq 0 \quad \text{for } x \in \ker T' \quad \text{then } x \in \text{ran } T.

For such an \( x' \), define the linear functional \( \ell \) on \text{ran } T by

\ell(y) = x'(x)

for every \( x + y = Tx \). The functional is well-defined

for \( \forall \quad T x_1 = T x_2 \), then \( x_1 - x_2 \in \ker T \) and hence

\( x'(x_1 - x_2) = 0 \). As (45.5) \implies (45.4) we know that \text{ran } T is closed and hence by (29.7) we can always chose \( x \in X \) so that \( c \| x \| \leq \| y \| \)

for some \( 0 < c < \infty \), independent of \( x \) and \( y \). Thus

\| \ell(y) \| = \| x' \| \| x \| \leq \frac{c}{\| x' \|} \| y \| \)

and so \( \ell \) is a bounded linear functional on \text{ran } T.
= \text{ran} T \quad \text{Let } y' \text{ be an extension of } l \text{ to } Y.\

Then for all } x \in X, \quad \langle T'y', x \rangle = \langle y', Tx \rangle = \langle y', x \rangle, \quad \text{and so } T'y = x' \quad \text{if } x' \in \text{ran} T'.\n
This completes the proof of Banach's Theorem (45.3).

\underline{Lecture 5} \quad \underline{Remark 57.1} \quad \text{Suppose } T \in L(X, Y), \quad \text{and hence } T' \in L(Y', X').\n
\hspace{1cm} \text{has closed range. It is interesting to apply (46.1) to } T'. \quad \text{We have}

\hspace{1cm} (57.2) \quad \text{ran } T' = \{ x' \in X' : \langle x'', x' \rangle = 0 \quad \forall \ x'' \in \ker T'' \}

\hspace{1cm} \text{But (46.2) also applies, so we also have}

\hspace{1cm} (57.3) \quad \text{ran } T' = \{ x' \in X' : \langle x', x \rangle = 0 \quad \forall \ x \in \ker T' \}

\hspace{1cm} \text{Using the injection } \varphi : X \rightarrow X', \quad \text{sending } x \mapsto x'' = \varphi(x), \quad x''(x') = x'(x). \quad \text{(see (26.3))}, \quad \text{we see that}

\hspace{1cm} \langle x'', x' \rangle = 0

\hspace{1cm} \text{for all } x'' \in \ker T'', \quad y \text{ and only if}

\hspace{1cm} \langle x'', x' \rangle = 0

\hspace{1cm} \text{for all } x'' \in X \equiv \ker T' \cap \text{ran } \varphi. \quad \text{But in general } M \not\subset \ker T''.