and M a closed, convex, convex, balanced subset of X. Then for any $x_0 \notin M$, $\exists$ a bounded linear functional x' on X such that

(28.1) $$x'(x_0) > 1 \quad \text{and} \quad |x'(x)| \le 1 \quad \text{for } x \in M.$$

Lecture 3

If X and Y are Banach spaces and $T \in \mathcal{L}(X,Y)$ is a bijection, then it follows immediately from the open mapping theorem that T^{-1}, the inverse of T, is bounded from Y onto X. In particular if $T \in \mathcal{L}(X,Y)$ is injective, then

(28.2) $$\operatorname{ran} T \text{ is closed} \iff \exists\, 0 < c < \infty \text{ st } c\|x\| \le \|Tx\|$$

If $T \in \mathcal{L}(X,Y)$, then, as noted before, $\ker(T) = \{x : Tx = 0\}$ is closed and hence the quotient space $X/\ker(T)$ is a Banach space. The map T induces a (well-defined) map $[T] \in \mathcal{L}(X/\ker(T), Y)$ according to the prescription

(29.1) $\quad [T][x] = T\hat{x} \quad$ for any $\hat{x} \in [x] = x + \ker(T)$

An elementary calculation (exercise) shows that

(29.2) $\quad \|[T]\| = \|T\|$

Now $[T]$ is injective from $X \setminus \ker(T)$ onto $\operatorname{ran}[T] = \operatorname{ran} T$ and hence by (28.2) we have the following equivalences.

<u>Theorem 29.3</u>

(29.4) $\quad \operatorname{ran} T$ is closed

(29.5) $\quad \exists\, 0 < c < \infty$ st $\quad c\|[x]\| \leq \|[T][x]\| \quad \forall\, [x] \in X \setminus \ker T$.

(29.6) $\quad \exists\, 0 < c < \infty$ with the following property; for any $x \in X$, $\exists$ element $u \in \ker(T)$ st $\quad c\,\|x + u\| \leq \|T(x+u)\|$

(29.7) $\quad \exists\, 0 < c < \infty$ with the following property: for any $y \in \operatorname{Ran} T$, $\exists\, x \in X$ st $y = Tx$ and $c\|x\| \leq \|y\|$

<u>Proof</u>: Exercise

We now give some examples of applications of the uniform boundedness principle.

(30.1) Exple: Suppose $\{T_n\}_{n=1}^{\infty}$ is a countable sequence of bounded linear operators from a Banach space X to a Banach space Y, such that for each $x \in X$

(30.2) $$x_\infty \equiv \lim_{n\to\infty} T_n x \qquad \text{exists}$$

Then

(30.3) $$\|T_n\| \le c \qquad n = 1, 2, \ldots$$

Indeed, $\exists\ N$ st $n \geq N \implies$

$$\|T_n x - x_\infty\| \le 1$$

In particular for $n \geq N$, $\|T_n x\| \le \|T_n x - x_\infty\| + \|x_\infty\|$
$$\le 1 + \|x_\infty\|$$

But $\|T_1 x\|, \ldots, \|T_{N-1} x\|$ is a finite set. Thus (30.3) follows

Note that if we replace n by a continuous variable in (30.2), i.e. for each $x \in X$, $x_\infty = \lim_{t\to\infty} T_t x$ exists, then $\sup_t \|T_t\|$ may not be bded.

(31.0) Exercise Show that if $T_n \in \mathcal{L}(\mathcal{H})$, $n \geq 0$, for some Hilbert space $\mathcal{H}$, and $\lim_{n\to\infty} (T_n x, y)$ $\exists$ $\forall x, y \in \mathcal{H}$, then $\|T_n\| \leq c$, $\forall n$. 31

(31.1) Exple 2

For $f \in L^1(0, 2\pi)$, define the finite Fourier series

(31.2)
$$S_n f(t) = \sum_{j=-n}^{n} \hat{f}(j) e^{ijt}$$

where

(31.3)
$$\hat{f}(j) = j^{th} \text{ Fourier coefficient of } f$$
$$= \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{ijt}\, dt$$

General question: Is $f(t) = \lim_{n\to\infty} S_n f(t)$ in some sense?

The answer is "yes" in some appropriate sense, (for example, in $L^2(0, 2\pi)$), but even if $f \in C([0, 2\pi])$, the convergence may not be pointwise. For example, $\exists f \in C([0, 2\pi])$ such that

(31.4)
$$S_n f(0) \not\to f(0)$$

Indeed, for $f \in C([0, 2\pi])$, set
$$\lambda_n(f) = (S_n f)(0)$$

As $|\lambda_n(f)| \leq \sum_{-n}^{n} |\hat{f}(j)| \leq \frac{1}{2\pi} \sum_{-n}^{n} \int_0^{2\pi} |f(t)|\, dt = \frac{2n+1}{2\pi} \int_0^{2\pi} |f|\, dt$
i.e. $|\lambda_n(f)| \leq 2n+1\ \|f\|_\infty$, it is clear that $\lambda_n \in (C[0,1])'$ for

each n.

If $S_n f(0) \to f(0)$ for every $f \in C([0,1])$, then by the principle of uniform boundedness $\exists\ 0 < c < \infty$, indep of n st

(32.1) $$|\lambda_n(f)| \leq c\, \|f\|_\infty \qquad \forall f \in C[0,1]$$

But

$$\lambda_n(f) = \sum_{-n}^{n} \hat{f}(j)\, e^{ij0}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x) \sum_{j=-n}^{n} e^{-ijx}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} f(x)\ D_n(u)$$

where

(32.2) $$D_n(u) = D_n(-u) = \sum_{-n}^{n} e^{iju}$$

$$= e^{-inu} \ \frac{1 - e^{i(2n+1)u}}{1 - e^{iu}}$$

$$= \sin\left(n + \tfrac{1}{2}\right)u \ / \sin u/_2$$

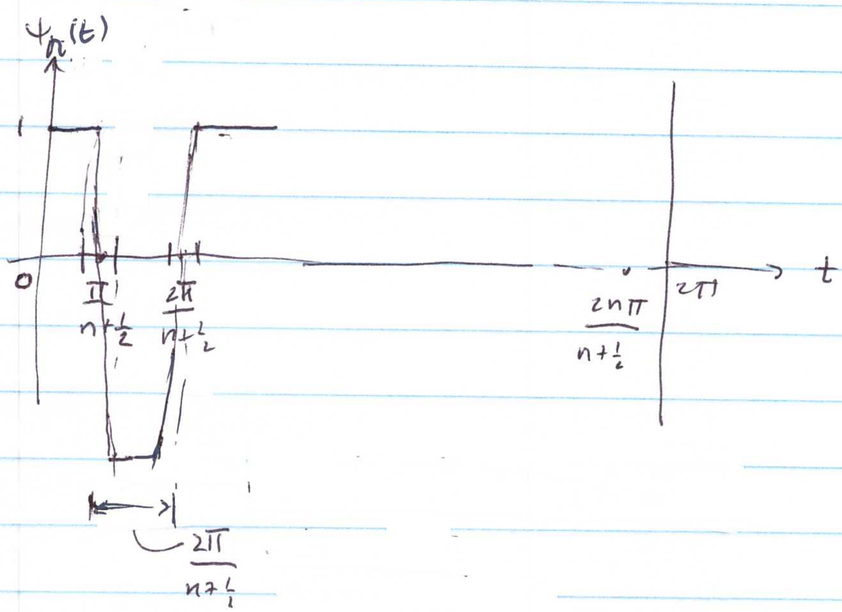
Now for $0 < u < 2\pi$

$$D_n(u) = 0 \iff \left(n + \tfrac{1}{2}\right)u = k\pi, \quad 1 \leq k \leq 2n$$

Let $\psi_n \in C([0, 2\pi])$ st

(33.1) $\|\psi_n\|_\infty = 1$

(33.2) $\psi_n(t) = \operatorname{sgn} D_n(t)$ except in a neighbourhood of total length $\varepsilon/2n$ about the points $\{ \frac{k\pi}{n+\frac{1}{2}}, 1 \le k \le 2n \}$



Then
$$|\lambda_n(\psi_n)| = \left| \frac{1}{2\pi} \int_0^{2\pi} \psi_n(x) D_n(x)\, dx \right|$$
$$\geq \frac{1}{2\pi} \left| \int_0^{2\pi} \operatorname{sgn} D_n(x)\, D_n(x) \right|$$
$$- \frac{1}{2\pi} \int_0^{2\pi} |\psi_n(x) - \operatorname{sgn} D_n(x)|\, |D_n(x)|\, dx$$

But as $|D_n(x)| \le 2n+1$;

$$\frac{1}{2\pi} \int_0^{2\pi} |\psi_n(x) - \operatorname{sgn} D_n(x)|\, |D_n(x)|\, dx$$
$$\le \frac{1}{2\pi} \cdot 2 \cdot \frac{\varepsilon}{2n} \cdot (2n+1) < \frac{2\varepsilon}{\pi} < \varepsilon$$

Thus

$$|l_n(\psi_n)| \geq \frac{1}{2\pi}\int_0^{2\pi} |D_n(x)|\,dx - \varepsilon$$

$$= \frac{1}{\pi}\int_0^{\pi} |D_n(x)|\,dx - \varepsilon, \text{ as } D_n(u) = D_n(2\pi - u)$$

$$\geq \frac{1}{\pi}\sum_1^{n-1}\int_{\frac{j\pi}{n+\frac{1}{2}}}^{(j+1)\pi/n+\frac{1}{2}} \left|\frac{\sin(n+\frac{1}{2})t}{\sin t/2}\right| dt + O(1)$$

$$\geq \frac{2}{\pi}\sum_1^{n-1}\int_{\frac{j\pi}{n+\frac{1}{2}}}^{\frac{(j+1)\pi}{n+\frac{1}{2}}} \left|\frac{\sin(n+\frac{1}{2})t}{t}\right| dt + O(1)$$

$$\geq \frac{2}{\pi}\sum_1^{n-1}\frac{n+1/2}{(j+1)\pi}\int_{\frac{j\pi}{n+\frac{1}{2}}}^{(j+1)\pi/n+\frac{1}{2}} |\sin(n+\tfrac{1}{2})t|\,dt + O(1)$$

$$= \frac{2}{\pi}\sum_1^{n-1}\frac{1}{(j+1)\pi}\int_{j\pi}^{(j+1)\pi} |\sin t|\,dt + O(1)$$

$$= \frac{4}{\pi^2}\sum_1^{n-1}\frac{1}{j+1} + O(1)$$

$$\sim \log n \to \infty$$

which contradicts (32.1). We conclude that $\exists f \in C[0,1]$

s.t. $S_n f(0) \not\to f(0)$ (in fact, s.t. $S_n f(0)$ does not converge to any number as $n \to \infty$.)

The open mapping theorem is equivalent to the

__Closed Graph Theorem__. Let T be a mapping from a space X to a space Y. Then

$$\Gamma(T) = \text{graph of } T = \{\langle x, y\rangle : \langle x,y\rangle \in X \times Y,\ y = Tx\}$$

__Th^m 35.1__ (Closed Graph Theorem)

Let X and Y be Banach spaces and T a linear map from X into Y. Then

$$T \text{ is bounded} \iff \Gamma(T) \text{ is closed in } X \times Y$$

__Exercise__: $X \times Y = \{\langle x, y\rangle : x \in X, y \in Y\}$ is a Banach space with norm $\|\langle x, y\rangle\| \equiv \|x\| + \|y\|$.

__Proof of Th^m 35.1__: The implication $\Rightarrow$ is clear. Suppose $\Gamma(T)$ is closed. As $\Gamma(T)$ is linear and closed in $X \times Y$, it is a Banach space. Consider the continuous maps

$$\pi_1 : \Gamma(T) \to X \quad \text{taking } \langle x, Tx\rangle \longmapsto x$$

and

$$\pi_2 : X \times Y \to Y \quad \text{taking } \langle x, y \rangle \longmapsto y$$

Now π_1 is clearly a bounded bijection from $\Gamma(T)$ onto X and hence by the open mapping theorem π_1^{-1} is continuous and hence $\pi_2 \circ \pi_1^{-1} \in \mathcal{L}(X, Y)$. But

$$\pi_2 \circ \pi_1^{-1}(x) = \pi_2(\langle x, Tx \rangle) = Tx$$

Thus T is continuous and hence bounded.

We now show that, conversely, the Closed Graph Theorem implies the Open Mapping Theorem. Let $T \in \mathcal{L}(X, Y)$ be onto. We must show that T is an open mapping. Assume first that T is 1-1.

Suppose $y_n \to y$ and $T^{-1}y_n \to x$. We show that $T^{-1}y = x$. Thus $\Gamma(T^{-1})$ is closed and hence T^{-1} is bounded, and hence continuous. Thus if O is open in X, then $(T^{-1})^{-1}O = TO$ is open i.e. T is open.

But as T is bded, $y_n = T(T^{-1}y_n) \to Tx$. But $y_n \to y$.
and so $Tx = y$, or $x = T^{-1}y$, as desired.

If T is onto, but not 1-1, $[T]$ is a bounded bijection from $X \setminus \ker(T)$ onto Y. By the previous argument $[T]$ is an open mapping. But it is easy to see (exercise) that $\pi : x \mapsto [x]$ is an open mapping and so $T = [T] \circ \pi$ is an open mapping. Thus Closed G. Th^m $\Rightarrow$ Open Mapping Th^m. $\square$

The above results show that

(37.1) $$\text{Open mapping Th}^m \equiv \text{closed graph Th}^m.$$

Operators, such as differential operators, which are not defined everywhere on the space at hand, are of great interest. In particular, if X and Y are linear spaces, we are interested in linear maps $T : X \to Y$ with domains $\operatorname{Dom} T = D(T)$, which are linear subspaces of X, $D(T) \subset X$. Of greatest interest are the situations where $D(T)$ is dense in X. For example consider the differential operator

(38.1) $$Tf = \frac{df}{dx}$$

with dense domain

(38.2) $$D(T) = \{f \in C^1[0,1]\} \subsetneq L^2(0,1) = \mathcal{H}.$$

Clearly if $T: X \to Y$, X and Y Banach spaces, is a bounded linear map with $\mathrm{Dom}(T)$ dense in X,

$$\|Tx\| \leq c\|x\| \quad x \in \mathrm{Dom}(T),$$

then T extends uniquely to a bded map on X with the same bound. (Check this!).

If $T: X \to Y$ is continuous, then

(38.1) $$X \ni x_n \to x \implies Tx_n \to y = Tx.$$

We say that a densely defined operator $T: X \to Y$ is <u>closed</u> if

(38.2) $$D(T) \ni x_n \to x \quad \underline{\text{and}} \quad Tx_n \to y$$

then

$$x \in D(T) \quad \text{and} \quad Tx = y.$$

Clearly, if T is bdd, then T is closed.

Theorem 39.1

Suppose T is an everywhere defined, closed operator from $X \to Y$, X and Y Banach spaces. Then T is bounded.

Proof: Consider the graph of T,

$$\Gamma(T) = \{ \langle x, Tx \rangle : x \in D(T) = X \}.$$

Suppose $x_n \to x$ and $Tx_n \to y$. Then, by definition, we certainly have $x \in D(T)$, and $Tx = y$ as T is closed. Hence $(x, y) \in \Gamma(T)$ and so $\Gamma(T)$ is closed which implies that T is bounded.

Exer: Show that T in (38.1) is closed on the extended domain $\tilde{D}(T)$,

$$\tilde{D}(T) = \{ f \in L^2(0,1) : f \text{ is absolutely continuous and } f' \in L^2(0, 2\pi) \}$$

Note that T on $\tilde{D}(T)$ is not bounded and this is consistent with the fact that $\tilde{D}(T) \subsetneq L^2(0,1)$.

Remarks 40.1

The proof of the Hahn - Banach Theorem follows from, and in fact, is equivalent to Zorn's Lemma. The proof of the open mapping theorem, and also the principle of uniform boundedness, follows from the Baire Category Theorem which asserts, in particular, that an infinite dimensional Banach space cannot be written as a countable union of nowhere dense sets. Thus if $X = \bigcup_{i=1}^{\infty} X_i$, then for some i, the closure $\overline{X_i}$ of X_i, must contain an open ball.

The dual $T' : Y' \to X'$ of an operator $T \in \mathcal{L}(X, Y)$ is given by

(40.1) $$\langle T'y', x \rangle = \langle y', Tx \rangle$$

i.e.

$$T'y'(x) = y'(Tx)$$

$\forall x \in X, \ y' \in Y'$. Clearly $T' \in \mathcal{L}(Y', X')$ and

(40.2) $$\|T'\| \le \|T\|$$

On the other hand, by (24.3),

$$\|Tx\| = \sup_{\|y'\| \le 1} |y'(Tx)| = \sup_{\|y'\| \le 1} |T'y'(x)|$$

$$\le \sup_{\|y'\| \le 1} \|T'y'\| \|x\|$$

$$\le \sup_{\|y'\| \le 1} \|T'\| \|y'\| \|x\| = \|T'\| \|x\|$$

and so $\|T\| \le \|T'\|$. Thus we have equality in (40.2)

(41.1) $$\|T'\| = \|T\|$$

Moreover the following is true:

Lecture 4 **Theorem 41.2** Let $T \in \mathcal{L}(X, Y)$. Then T is a bijection from $X \to Y$ if and only if T' is a bijection from $Y' \to X'$ and if T, or equivalently T', is a bijection, then $(T')^{-1} = (T^{-1})'$ and as $T^{-1} \in \mathcal{L}(Y, X)$, $(T')^{-1} \in \mathcal{L}(X', Y')$.

Proof: Indeed if T is a bijection, then unravelling the definition we see immediately that

(41.3) $$\langle (T^{-1})' x', y \rangle = \langle x', T^{-1} y \rangle \qquad \forall x' \in X', y \in Y$$