of the set $\{(\lambda_1, \dots, \lambda_n) : \sum_{i=1}^{n} |\lambda_i| \le 1\}$ in $\mathbb{C}^n$.

Conversely suppose that the unit sphere in X is compact and that X contains an infinite set $\{x_i\}_{i\ge 1}$ of independent vectors. For each $n \ge 1$, let X_n be the subspace generated by $\{x_1, \dots, x_n\}$. Then $X_1 \subsetneq X_2 \subsetneq X_3 \subsetneq \cdots \subsetneq X_n \subsetneq X_{n+1} \subsetneq \cdots$ is a strictly ascending chain of closed subspaces in X. By Theorem 14.1, for each $n \ge 1$, there exists $\hat{x}_n \in X_{n+1}$ such that

$$\|\hat{x}_n\| = 1 \quad \text{and} \quad \operatorname{dist}(\hat{x}_n, X_n) \ge \frac{1}{2}$$

But then ~~for~~ $k > \ell$, $\|\hat{x}_k - \hat{x}_\ell\| \ge \operatorname{dist}(\hat{x}_k, X_k) \ge \frac{1}{2}$ which contradicts the compactness of the unit sphere in X. This proves the Corollary. $\square$.

Lecture 2

Basic operator Theory

An operator T from a linear space X to a linear space Y is linear if

(17.1) $$T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T x_1 + \lambda_2 T x_2$$

for all $x_1, x_2 \in X$, $\lambda_1, \lambda_2 \in \mathbb{C}$.

If a linear operator T is a bijection from X onto Y, we say that X is isomorphic to Y, written $X \cong Y$, and T provides the isomorphism.

Clearly if $X \cong Y$, then $\dim X = \dim Y$.

A linear operator T from a Banach space X to a Banach space Y is bounded if there exists a constant c, $0 < c < \infty$, such that

(17.2) $$\|Tx\|_Y \leq c \|x\|_X \quad \forall x \in X$$

The space of bounded linear operators from X to Y, denoted by $\mathcal{L}(X, Y)$, is (exercise) a Banach space in its own right with norm

(18.1) $$\|T\| \equiv \sup_{\|x\|_X \le 1} \|Tx\|_Y$$

It is easy to see (exercise) that the following statements are equivalent for a linear operator T from $X \to Y$.

(18.2) T is bounded

(18.3) T is continuous

(18.4) T is continuous at $x = 0$

The null space of T, $\{x : Tx = 0\}$ is a linear space and is denoted by $\ker(T)$, or $\mathrm{nul}(T)$, and $\mathrm{Ran}\,T$, also a linear space, denotes the range of T. If $X = Y$, we write $\mathcal{L}(X)$ for $\mathcal{L}(X, X)$. A linear functional x' on X is a linear map from X to $Y = \mathbb{C}$. The dual space of X, denoted X', is the span of all bounded linear functionals on X. Thus $X' = \mathcal{L}(X, \mathbb{C})$

<u>Exercise:</u> Every linear map $f: X \to Y$ is bounded iff $\dim X < \infty$.

If $T: X \to Y$ is bounded, then $\ker T$ is a closed subspace of X. $\operatorname{Ran} T$, however, may not be closed (exercise).

Examples of dual spaces (see [Yosida]).

- Let $(M, \mathcal{a}, \mu)$ be a measure space that is σ-finite ie $\exists\, A_n \in \mathcal{a}$, $\mu(A_n) < \infty$, such that $M = \bigcup_n A_n$.

(19.1) Then $(L^p(M, \mathcal{a}, \mu))' = L^q(M, \mathcal{a}, \mu)$ (i.e. the spaces are <u>isometrically isomorphic</u> viz $\exists$ a linear map T from one space onto the other which preserves norms) if $1 \le p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. In particular for the counting measure μ on the positive integers, $M = \{1, 2, 3, \dots\}$ we have

(19.2) $$(\ell^p)' = \ell^q \quad , \ 1 \le p < \infty$$

where

(20.1) $$\ell^p \equiv \{ x = (x_1, x_2, \dots) : \sum_1^\infty |x_i|^p < \infty \}$$

and

(20.2) $$\ell^\infty = \{ x : (x_1, x_2, \dots) : \sup_i |x_i| < \infty \}$$

(20.3) Let $c = \{ x : (x_1, x_2, \dots) : \lim_{i\to\infty} x_i \ \exists \} \subset \ell^\infty$

with norm $\|x\| = \sup_i |x_i|$

(20.4) $$c' = \ell^1$$

(20.5) Let $c_0 = \{ x = (x_1, x_2, \dots) : \lim_{i\to\infty} x_i = 0 \} \subset c$

Then again

$$c_0' = \ell^1$$

Now $c_0 \neq c_1$ i.e. they are not isometrically isomorphic — see Exercises. Thus we see in particular that it is possible that $X_1' = X_2'$ but $X_1 \neq X_2$.

Insert 20+1 →

(20.6) $(L^\infty(M, \mathcal{a}, \mu))' \neq L^1(M, \mathcal{a}, \mu)$ and is given by the so-called finitely additive measures (see [Yosida]). The fact

Insert on p20

Note that it follows from the polarization identity

(20+1.1) $$(u,v) = \frac{1}{4}\left[\left(\|u+v\|^2 - \|u-v\|^2\right) - i\left(\|u+iv\|^2 - \|u-iv\|^2\right)\right]$$

that if T is an isometric isomorphism between two Hilbert spaces $(\mathcal{H}, (\cdot,\cdot)_{\mathcal{H}})$ and $(\mathcal{K}, (\cdot,\cdot)_{\mathcal{K}})$, then T is automatically a <u>unitary map</u> from $\mathcal{H}$ onto $\mathcal{K}$,

(20+1.2) $$(Tx, T\hat{x})_{\mathcal{K}} = (x, \hat{x})_{\mathcal{H}}, \quad x, \hat{x} \in \mathcal{H}$$

In other words, the spaces are <u>unitarily equivalent</u>.

Another way to state (20+1.2) is that if $T : \mathcal{H} \to \mathcal{K}$ preserves sizes, then it automatically also preserves angles (the angle θ between x and $\hat{x}$ is defined to be $\cos\theta = |(x,\hat{x})| / \|x\|\,\|\hat{x}\|$.)

that $(L^\infty)' \neq L^1$ follows abstractly from the fact that 'if X' is separable, then X is separable.

(21.1) Let S be a compact topological space and let $C(S)$ denote the continuous function on S with $\|f\|_\infty = \sup_{x\in S} |f(x)|$. Then $(C(S))'$ is given by the complex Baire measures on S as follows: $x' \in (C(S))' \iff x'(f) = \int_S f(s)\,d\mu(s)$ for some complex Baire measure with $\|x'\| = \sup_{\|f\|_\infty \le 1} \left| \int f(s)\,d\mu(s) \right| < \infty$.

We will often use the pairing $\langle \cdot, \cdot \rangle$ to denote the action of x' on X. Thus

(21.2)
$$\langle x', x \rangle = x'(x) \quad \text{for } x' \in X,\ x \in X.$$

If Y is a subspace of a linear space X, then X/Y denotes the quotient space with elements that are the cosets $[x] = x + Y$, $x \in X$. If X is a Banach space and Y is a closed subspace, then $X \setminus Y$ is also a

Banach space with norm (exercise)

(22.1) $$\|[x]\| = \inf_{u \in Y} \|x+u\|$$

A simple argument (exercise) shows that if π denotes the map $x \mapsto [x] = x + Y$ taking X onto $X \setminus Y$, then

(22.2) $$\|\pi\| = 1$$

Many basic results in Functional Analysis are consequences of the following three results; the Hahn-Banach extension theorem, the open mapping theorem, and the principle of uniform boundedness. For proofs, see [Yosida] [Simon] [Lax].

Theorem 22.3 (Hahn-Banach extension theorem) Let X be a linear space and p a semi-norm defined on X. Let M be a linear subspace of X and f a linear functional defined on M such that $|f(x)| \le p(x)$ on M. Then $\exists$ a linear functional F defined on X such that

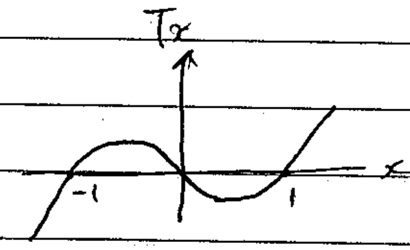
(23.1) F is an extension of f, i.e. $F(x) = f(x)$ for $x \in M$, and

(23.2) $|F(x)| \le p(x)$ on X

Theorem 23.3 (Open mapping Theorem) Let T be a bounded linear map from a Banach space X onto a Banach space Y. Then T is an open mapping i.e. $TS = \{T(x) : x \in S\}$ is open in Y if S is open in X.

Remark 23.4 Theorem 23.3 relies on the linearity of T.

For example $Tx = (x^2 - 1)x$



clearly maps $\mathbb{R}$ continuously onto $\mathbb{R}$, but $T(0,1)$ is not open although $(0,1)$ is open.

Theorem 23.5 (Principle of uniform boundedness)
Let $\{T_a : a \in A\}$ be a family of bounded linear operators defined on a Banach space X into a Banach space Y. Then the boundedness of $\{\|T_a x\| : a \in A\}$

for each $x \in X$ implies the boundedness of $\{ \|T_a\| : a \in A \}$

If X is a Banach space, then $p(x) = \|x\|$ defines a semi-norm on X. Fix $x_0 \neq 0$ in X: then

(24.1) $$f(\lambda x_0) = \lambda \|x_0\|$$

defines a linear functional on the subspace $M = \{\lambda x_0 : \lambda \in \mathbb{C}\} \subset X$ with $|f(x)| = \|x\| = p(x)$, $x \in M$. By the Hahn-Banach theorem f extends to a linear functional $F(x)$ on X such that $|F(x)| \leq p(x) = \|x\|$, $x \in X$. Thus f extends to a bounded linear functional $F \in X'$, $\|F\| = 1$. This shows, in particular, the following:

(24.1) For any $x_0 \in X$, $\exists$ $F \in X'$, $\|F\| = 1$ such that $F(x_0) = \|x_0\|$ and so

(24.2) X' is non trivial

(24.3) For any $x \in X$
$$\|x\| = \sup_{\{x' \in X' : \|x'\| \leq 1\}} |x'(x)|$$

This result is dual to the fact that for any $x' \in X'$

(25.1) $$\|x'\| = \sup_{\|x\| \leq 1} |x'(x)|$$

More generally if V is a closed subspace of X, and $0 \neq x_0 \in X \setminus V$ then

(25.2) $$f(\lambda x_0 + v) = \lambda \|x_0\| \quad , \quad \lambda \in \mathbb{C} \quad , \quad v \in V$$

defines a linear function on the subspace

$$M = \{\lambda x_0 + v : \lambda \in \mathbb{C}, v \in V\}$$

Moreover, for $\lambda \neq 0$,

$$\|\lambda x_0 + v\| = |\lambda| \, \|x_0 + v/\lambda\| \geq |\lambda| d_0$$

where

$$d_0 = \operatorname{dist}(x_0, V) > 0 .$$

Hence

$$|f(\lambda x_0 + v)| = |\lambda| \|x_0\| \leq \frac{\|x_0\|}{d_0} \|\lambda x_0 + v\|$$

and so f is bounded on M. By the Hahn-Banach theorem f extends to a bded lin. functional x' on X with $\|x'\| \leq \|x_0\|/d_0$. We conclude, in particular, the following:

<u>Proposition 26.1</u> If V is a closed subspace of X and $x_0 \notin V$, then $\exists\, x' \in X'$ st

(26.2) $$x'(x_0) = \|x_0\| \quad \text{and} \quad x'(v) = 0 \text{ for all } v \in V.$$

A Banach space X inbeds naturally into its <u>double dual</u> $X'' = (X')'$ via the map

$$X \ni x \mapsto \varphi(x) \in X''$$

where

(26.3) $$\varphi(x)(x') \equiv x'(x)$$

By (25.1)

$$\|\varphi(x)\| = \sup_{\|x'\| \le 1} |\varphi(x)(x')| = \sup_{\|x'\| \le 1} \|x'(x)\| = \|x\|$$

and so the imbedding φ is in fact isometric. A space X is <u>reflexive</u> if φ is surjective ie $X'' \cong X$. Generally, however, $\varphi(X) \subsetneq X''$. In particular we see from (19.1) and (20.6) that $L^p(M, \mathcal{A}, \mu)$ is reflexive for

$1<p<\infty$ but not for $p=1$. ~~All~~ Hilbert spaces are reflexive: indeed, by the Riesz representation theorem for any $x' \in X'$, $\exists$ a unique $x \in \mathcal{H}$ such that $x'(y) = (x, y)$ for all $y \in \mathcal{H}$ and $\|x'\| = \|x\|$. Thus $X \cong X' \cong (X')' \cong X''$; moreover the mapping $X \to X''$ is isometric.

We denote by φ the above map $x \mapsto x'$ taking $\mathcal{H}$ to $\mathcal{H}'$, $x'(y) = (x, y)$ for all $y \in \mathcal{H}$. The map φ is anti-linear ie $\varphi(\lambda x_1 + \mu x_2) = \bar{\lambda}\varphi(x_1) + \bar{\mu}\varphi(x_2)$, $x_1, x_2 \in \mathcal{H}$ and $\lambda, \mu \in \mathbb{C}$.

A more refined version of the argument leading to (24.3) yields the following (special case of a) theorem of S. Mazur. Recall that a subset M of a linear space X is balanced if $x \in M$ and $|\lambda| \leq 1$, then $\lambda x \in M$.

Theorem 27.1 (see [Yosida]) Let X be a Banach space and M a closed, convex, convex, balanced subset of X. Then for any $x_0 \notin M$, $\exists$ a bounded linear functional x' on X such that

(28.1) $$x'(x_0) > 1 \quad \text{and} \quad |x'(x)| \leq 1 \quad \text{for } x \in M.$$

Lecture 3

If X and Y are Banach spaces and $T \in \mathcal{L}(X, Y)$ is a bijection, then it follows immediately from the open mapping Theorem that T^{-1}, the inverse of T, is bounded from Y onto X. In particular if $T \in \mathcal{L}(X, Y)$ is injective, then

(28.2) $$\operatorname{ran} T \text{ is closed} \iff \exists\, 0 < c < \infty \text{ st } c\|x\| \leq \|Tx\|$$

If $T \in \mathcal{L}(X, Y)$, then, as noted before, $\ker(T) = \{x : Tx = 0\}$ is closed and hence the quotient space $X/\ker(T)$ is a Banach space. The map T induces a (well-defined) map $[T] \in \mathcal{L}(X/\ker(T), Y)$ according to the