we can conclude that
\[
\sum_{n=1}^{N} \langle \xi_n, (A + B^*) \xi_n \rangle \leq \|A\| + \|B\| < \infty
\]
and thus $A + B \in B_C$. But to prove (16.1), we need only to prove that
\[
\|U^* V A V^* U - \|A\| < \epsilon.
\]

Picking an orthonormal basis $\{\psi_n\}$ with each $\psi_n$ in $\ker U$ or $\ker (\ker U)^{\perp}$, we see that
\[
\|U^* V A V^* U - \|A\| = \sum \langle \xi_n, U^* V A V^* U \xi_n \rangle
\]
\[
\leq \sum \langle U \psi_n, V(AV^* U) \psi_n \rangle
\]
\[
\leq \|U^* V A V^* U - \|A\|.
\]

Similarly picking an orthonormal basis $\{\psi_n\}$ with each $\psi_n$ in $\ker V^*$ or $(\ker V^*)^\perp$, we find
\[
\|V A V^* - \|A\| < \epsilon.
\]

Lecture 13 (b) By the lemma proven below, each $B \in \mathcal{L}(C^4)$ can be written as a linear combination of 4 unitary operators and
so by (a) we only need to show that \( A \in B_1 \),

\[
1 \leq U A \in B, \text{ and } A U \in B, \text{ if } U \text{ is unitary.}
\]

But \( |U A| = \sqrt{(U A)^* U A} = \sqrt{A^* A} = |A| \)

and \( (U^* A U) (U^* A U) = U^* |A|^2 U \\
= U^* A^* A U \\
= (A U)^* A U = |A U|^2 \)

and so \( |A U| = U^{-1} |A| U \) and no by part (c) of \( \text{Thm 1.58.1, } A U \text{ and } U A \in B_1. \)

(c) Let \( A = U |A| \) and \( A^* = V |A^*| \) be the polar decompositions of \( A = A^* \). Then \( A^* = |A| U^* \)

and \( |A^*| = V^* V |A^*| = V^* A^* = V^* |A| U^* \)

If \( A \in B_1 \), then \( |A| \in B_1 \) and so \( |A^*| \in B_1 \), by (b),

and no \( A^* = V |A^*| \in B_1 \), again by (b).

\[\square\]

\textbf{Lemma (163.1)} Every \( B \in \mathcal{L}(H) \) can be written as a linear combination of 4 unitary operators

\textbf{Proof:} Since \( B = \frac{1}{2} (B + B^*) - \frac{i}{2} (i (B - B^*)) \), \( B \) can be
written as a linear combination of 2 self-adjoint operators. So, suppose \( A \) is self-adjoint and without loss we can assume that \( \|A\| = 1 \). Then

\[ A = \frac{1}{2} (A + i\sqrt{1-A^2}) + \frac{1}{2} (A - i\sqrt{1-A^2}) \]

are unitary and

\[ A = \frac{1}{2} (A + i\sqrt{1-A^2}) + \frac{1}{2} (A - i\sqrt{1-A^2}) \]

\[ A \text{ unitary} \quad A = \frac{1}{2} (A + i\sqrt{1-A^2}) + \frac{1}{2} (A - i\sqrt{1-A^2}) \]

**Qm 164.1**

Let \( \| \cdot \| \) be defined in \( B_1 \) by \( \| A \| = \| A \| \).

Then \( B_1 \) is a Banach space with norm \( \| \cdot \| \), i.e.,

\[ \| A \| = \| A \| \]

**Proof:** Exercise.

**Exercise 164.2**

Show that if \( A \in B_1(\mathbb{H}) \) and \( B \in L(\mathbb{H}) \) then

\[ \| BA \| \leq \| B \| \| A \| , \]

\[ \| AB \| \leq \| B \| \| A \| , \]

(Hint: Use min-max principle.)
Theorem 165.1

Every $A \in \mathcal{B}_1$ is compact. A compact operator $A$

is in $\mathcal{B}_1$ if and only if $\sum_{n=1}^{\infty} \lambda_n < \infty$ where $\lambda_n$ are

the singular values of $A$.

Proof: Since $A \in \mathcal{B}_1$, $||A||^2 = \|A^*A\| \leq 2\|A\|^2$ and so

$$\|A\|^2 = \sum_{n=1}^{\infty} \|A\| \|\psi_n\|^2 < \infty$$

for any orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$. Suppose $y \in \text{span}\{\psi_1, \ldots, \psi_m\}$ and $\|y\| = 1$, then we have

$$\|Ax\|^2 \leq \|A\|^2 \|x\|^2 - \sum_{n=1}^{m} \|A\| \|\psi_n\|^2$$

Since $\{\psi_1, \ldots, \psi_m, \psi\}$ can always be completed to an orthonormal

basis. Thus as $m \to \infty$

$$\sup \{\|Ax\| : x = \psi_1, \ldots, \psi_m \} \to 0$$

Thus

$$\left( A - \sum_{n=1}^{N} \langle \psi_n, \cdot \rangle A \psi_n \right) \psi = 0 \quad \text{if} \quad \psi \in \text{span}\{\psi_1, \ldots, \psi_N\}$$

and

$$\sum_{n=1}^{N} \langle \psi_n, \cdot \rangle A \psi_n \to A \text{ in norm and so } A \text{ is compact.}$$
Finally, as

\[(166.1)\] 

\[1A_1 = \sum n (u_n, 1_u_n) \]

where \( u_n \) are the singular values of \( 1A_1 \), and

\[1A_1 u_n = \lambda_n u_n = \lambda_n u_n, \quad n \rightarrow 0, \quad \text{we have} \]

\[1 - 1A_1 = \sum (u_n, 1A u_n) = \sum n \]

as we can always include the eigenvectors of \( 1A_1 \)

corresponding \( u_n = 0 \) in \( (166.1) \).

**Corollary 166.2**

The finite rank operators are \( \| 1 \|_1 \)-dense in \( B_1 \).

The second class of operators which we will discuss on a separable Hilbert space \( \mathcal{H} \) are the Hilbert–Schmidt operators. This is a class of operators in \( L^2(\mathcal{H}) \) itself a Hilbert space.

**Definition 166.3.** An operator \( T \in L(\mathcal{H}) \) is called Hilbert–Schmidt if and only if \( \| T^* T \| = \| T \|^2 < \infty \).

The family of all Hilbert–Schmidt operators is denoted by \( B_2 \).
Th 167.1

(a) \( B_2 \) is a \( \mathcal{K} \)-ideal.

(b) If \( A, B \in B_2 \), then for any orthonormal basis \( \{ e_n \}_{n=1}^{\infty} \),

\[
\sum_{n=1}^{\infty} (\langle e_n, A^* B (e_n) \rangle)
\]

is absolutely summable, and its limit, denoted by \((A, B)_2\), is independent of the orthonormal basis chosen.

(c) \( B_2 \) with inner product \((A, B)_2\) is a Hilbert space.

(d) If \( \|A\|_2 = \sqrt{(A, A)_2} = \sqrt{\text{tr}(A^* A)} \), then

\[
\|A\| \leq \|A\|_2 \leq \|A\|_1,
\]

and

\[
\|A\|_2 = \|A^*\|_2.
\]

(e) Every \( A \in B_2 \) is compact and a compact operator, \( A \), is in \( B_2 \) if and only if \( \sum_{n=1}^{\infty} \lambda_n^2 < \infty \) where \( \lambda_n \) are the singular values of \( A \).

(f) The finite rank operators are \( \|\cdot\|_2 \)-dense in \( B_2 \).
(g) \( A \in B_2 \) if and only if \( \| A \| \leq 1 \), for some orthonormal basis \( \{ e_n \} \).

(h) \( A \in B_1 \) if and only if \( A = B C \) with \( B, C \in B_2 \).

**Proof:** Exercise. Use arguments analogous to the case \( B_1 \). □

Note that at the technical level it is much easier to detect when an operator \( T \) is in \( B_2 \), rather than \( B_1 \). All we have to do is show that

\[
\sum_{n=1}^{\infty} \| T e_n \|^2 < \infty
\]

for some orthonormal basis \( \{ e_n \} \). The key point is that (168.1) requires a calculation involving only \( T \) and its operator \( T \), which is given to us. However, to see if \( T \in B_1 \), we must show that

\[
\sum_{n=1}^{\infty} (e_n, T T e_n) < \infty
\]

for some orthonormal basis \( \{ e_n \} \). But we are making a
with $I_1$ which is a transcendental function of $T$:

$$I_1 = \sqrt{I_1 \ast T},$$

and not at all explicit.

This is why (h) above is no useful: to show that $T \in \mathcal{B}_1$, we just to show that $T$ is a product of 2 op's in $\mathcal{B}_2$, which is a non-transcendental task.

Moreover, if $\phi \in L^2(\mathcal{M}, \mu)$ for some measure space $(\mathcal{M}, \mathcal{A}, \mu)$, then $\mathcal{B}_2$ has a concrete realization.

\[ \text{Proposition 169.1} \]

Let $(\mathcal{M}, \mathcal{A}, \mu)$ be a measure space and separable, and let $\phi \in L^2(\mathcal{M}, \mu)$. Then $A \in L^2(\mathcal{H})$ is Hilbert-Schmidt if and only if there is a function

$$k \in L^2(\mathcal{M} \times \mathcal{M}, \mu \otimes \mu)$$

with

$$A \phi(x) = \int_{\mathcal{M}} k(x, y) \phi(y) \, d\mu(y), \quad \phi \in L^2.$$

Moreover,

$$\|A\|_2^2 = \int \int k(x, y) \, d\mu(x) \, d\mu(y).$$
Proof. Let \( K \in L^2(\mathcal{H},\mathcal{M},\mathcal{Q};\mathcal{Q}) \) and let \( A_k \) be the associated integral operator. It is easy to see (exercise!) that \( A_k \) is a well-defined operator on \( \mathcal{H} \) and that

\[
\|A\|_{L^2} \leq \|K\|_{L^2}
\]

Let \( \{\psi_n\}_{n=1}^{\infty} \) be an orthonormal basis for \( L^2(\mathcal{M},\mathcal{Q}) \). Then \( \{(\psi_n(x),\psi_m(y))\}_{n,m=1}^{\infty} \) is an orthonormal basis for \( L^2(\mathcal{H},\mathcal{H};\mathcal{Q};\mathcal{Q}) \).

(check this!) So

\[
K = \sum_{n,m} d_{nm} \psi_n(x) \overline{\psi_m(y)}
\]

Let

\[
K_N = \sum_{n,m=1}^{N} d_{nm} \psi_n(x) \overline{\psi_m(y)}
\]

Then each \( K_N \) is the integral kernel of a finite rank operator. In fact

\[
A_{K_n} = \sum_{n=1}^{N} d_{nm} (\psi_m,\psi_n)_Y
\]

Since \( \|K_N - K\|_{L^2} \to 0 \), \( \|A_k - A_{K_N}\| \to 0 \) as \( N \to \infty \) by (70.1). Thus \( A_k \) is compact and in fact
\[ \| A^k A_k \| = \sum_{n=1}^{\infty} \| A_k e_n \|_2^2 = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} | a_{nm} |^2 = \| k \|_2^2 \]

Thus \( A_k \in B_2 \) and \( \| A_k \|_2 = \| k \|_2 \).

We have shown that the map \( K \) is an isometry of \( L^2(M \times \mathbb{N}, \mu \otimes \mu) \) into \( B_2 \), so its range is closed. But the finite rank operators clearly come from kernels and since they are dense in \( B_2 \), the range of \( K \) is \( A_k \) is all of \( B_2 \). \( \square \)

Note that if an operator \( A \) has a kernel \( K \), then \( \| K \|_\infty < \infty \) is a very useful condition to verify that that \( A \) is compact.

We now finally define the trace of an operator.

**Theorem 171.1** If \( A \in B_1 \) and \( \{ e_n \}_{n=1}^{\infty} \) is any orthonormal basis, then \( \sum_{n=1}^{\infty} (\langle e_n, A e_n \rangle) \) converges absolutely and the limit is independent of the choice of basis.
Proof: We write $A = U^\perp A U^\perp + UA + A^\perp U^\perp$. Then
\[
((\phi_n, A \phi_n) | = \| U A U^\perp \phi_n \| \| U A^\perp \phi_n \|
\]
Thus
\[
\sum_{n=1}^{\infty} |(\phi_n, A \phi_n)| \leq \left( \sum_{n=1}^{\infty} \| U A U^\perp \phi_n \|^2 \right)^{1/2} \left( \sum_{n=1}^{\infty} \| U A^\perp \phi_n \|^2 \right)^{1/2}
\]
so since $1A U^\perp$ and $1A^\perp U$ are in $B_\infty$, the sum converges.

We have
\[
(172.1) \quad A = \sum_{n=1}^{\infty} \lambda_n (\phi_n, \phi_n) \phi_n \quad \text{when } \sum \lambda_n < \infty, \lambda_n > 0
\]
$\{\phi_n\}$, $\{\phi_n\}$ orthonormal sets, by (153.1).

Thus for any orthonormal basis $\{\phi_n\}$
\[
\sum_{n} |(\phi_n, A \phi_n)| = \sum_{n} \sum_{m} \lambda_m (\phi_m, \phi_n) (\phi_n, \phi_m)
\]
\[
= \sum_{m} \lambda_m \sum_{n} (\phi_m, \phi_n) (\phi_n, \phi_m)
\]
\[
= \sum_{m} \lambda_m (\phi_m, \phi_m)
\]
which is independent of $2\phi_m$.

Definition 173.1. The map

\[ \text{tr} : \mathcal{B}_i \rightarrow \mathbb{C} \]

given by \( \text{tr} A = \sum_{n=1}^{\infty} (e_n, A e_n) \) where \( \{e_n\} \)
is any orthonormal basis in called the trace.

It is not true (exercise!) that \( \sum_{n=1}^{\infty} |(e_n, A e_n)| < \infty \)

for some orthonormal basis implies that \( A \in \mathcal{B}_i \). For

\( A \) to be in \( \mathcal{B}_i \), it is necessary that the sum is finite for all orthonormal bases.

Here are properties of the trace.

**Theorem 173.1.**

(a) \( \text{tr}(\cdot) \) is linear.

(b) \( \text{tr} A^* = \overline{\text{tr} A} \)

(c) \( \text{tr} AB = \text{tr} BA \) if \( A \in \mathcal{B}_i \) and \( B \in \mathcal{B}_j \)

(d) \( |\text{tr} A| \leq \|A\| \)

**Proof:** (a) and (b) are obvious. To prove (c), use
(172.1) \[ A = \sum \lambda_m (f_m, e_n) e_n \]

\[ t^* A B = t^* \left( \sum \lambda_m (B^* f_m, e_n) e_n \right) \]

\[ = \sum_m (f_m, \left( \sum \lambda_n (B^* f_m, g_n) g_n \right) g_m) \]

as the \( g_n's \) are orthonormal and if \( g \perp g_m \) then \( \langle g, A B g \rangle = 0 \). Thus

\[ t^* A B = \sum_m (B^* f_m, g_m) = \sum \lambda_m (f_m, B g_m) \]

and

\[ t^* B A = t^* \left( \sum \lambda_n (f_n, B g_m) \right) \]

\[ = \sum_m (f_m, \left( \sum \lambda_n (f_n, B g_m) g_m \right) g_m) \]

\[ = \sum_m \lambda_m (f_m, B g_m) \]

and so \( t^* A B = t^* B A \). Finally

(c) \[ |t^* A| = \left| \sum \lambda_m (f_m, A e_n) \right| \]

\[ = \left| \sum_m \lambda_n \left( \sum_{m} \lambda_m (f_m, g_m) g_m \right) e_n \right| \]

Let \( \{ e_n \} \) be the completion of \( \{ g_m \} \) to an orthonormal basis.

Then \( |t^* A| \leq \sum \lambda_m = t^* \lambda A \)**
as \( |(\mathbf{f}_n, g_n)| \leq \|f_n\| \|g_n\| \leq 1 \). □

The following result show that \( B_i \) is the dual space of the compact operators \( K = K(\mathcal{H}) \) and the dual space of \( B_i \) is just \( L(\mathcal{H}) \).

\( \overline{175} \)

(a) \( B_i = (K(\mathcal{H}))' \), i.e., the map \( A \rightarrow \mathfrak{t}_A \) is an isometric isomorphism of \( B_i \) onto \( (K(\mathcal{H}))' \).

(b) \( L(\mathcal{H}) \supset (B_i)' \), i.e., the map \( B \rightarrow \mathfrak{t}(B) \) is an isometric isomorphism of \( L(\mathcal{H}) \) onto \( B_i' \).

In particular \( K(\mathcal{H}) \) is a non-reflexive Banach space.

Proof: See Reed-Simon, Vol I, Problem 30, Chapter VI. □

We are now in a position to define the determinant of an appropriate class of operators, viz.,

\[ \det (1 + A) \] where \( A \in B_i \).
Comment: We have introduced $B_1$ and $B_2$ and now that for $A \in B_1$, $\|A\|_1 = \sum \sigma n$ and for $A \in B_2$, $\|A\|_2 = \left( \sum \sigma n^2 \right)^{\frac{1}{2}}$, where the $\sigma n$ are the singular values of $A$. For general $p \geq 1$, we define the $p^{th}$ Schatten-class $\mathcal{B}_p$ as the class of compact operators $A$ with $\sum_{n=1}^{\infty} \lambda_n^p < \infty$.

$\mathcal{B}_p$ is a Banach space with norm $(\|A\|_p = \left( \sum \lambda_n^p \right)^{\frac{1}{p}}$.

For more details, e.g., B. Simon, Trace ideals and their applications.

Just as $\|A\|_1 = \sum \sigma n$ where the $\sigma n$ are the singular values of $A$ and hence the eigenvalues of $\|A\|_1$,

we expect that if $A \in B_1$, then

$$\|A\|_1 = \sum \sigma n$$

where the $\sigma n$ are the eigenvalues of $A$. This is true, but, as we will see, rather difficult to prove.
We are now in a position to define the determinant of an appropriate class of operators, viz.,
\[
\det (I + A)
\]
when \( A \in \mathcal{B}_1 \).

We need some alternating algebra, i.e., the theory of antisymmetric tensor spaces.

Given a Hilbert space \( \mathcal{H} \), \( \otimes \mathcal{H} \) is defined as the vector space of multilinear functionals on \( \mathcal{H} \). Explicitly, given \( \xi_1, \ldots, \xi_n \in \mathcal{H} \), we define \( \xi_1 \otimes \cdots \otimes \xi_n \in \otimes \mathcal{H} \) by
\[
(\xi_1 \otimes \cdots \otimes \xi_n, \eta_1, \ldots, \eta_n) = (\xi_1, \eta_1) \cdots (\xi_n, \eta_n)
\]
for \( \eta_i \in \mathcal{H} \), \( 1 \leq i \leq n \). It is an exercise (cf. Reed Simon Vol I, p49 et seq) that the finite span of the \( (\xi_1 \otimes \cdots \otimes \xi_n) \) possesses a well defined inner product with
\[(e_1 \otimes \cdots \otimes e_n, \eta_1 \otimes \cdots \otimes \eta_n)\]

is the completion of this finite span in the topology generated by this inner product. Given any \(A \in \mathcal{L}(\mathcal{H})\), there is a natural operator \(T_n(A)\) in \(\mathcal{L}(\otimes^n \mathcal{H})\) with

\[T_n(A)(e_1 \otimes \cdots \otimes e_n) = A e_1 \otimes \cdots \otimes A e_n\]

\(T_n\) satisfies

\[T_n(AB) = T_n(A) T_n(B)\]  \((178.1)\)

Let \(\mathcal{P}_n\) denote the group of all permutations on \(n\) letters. Let \(\varepsilon(\cdot)\) be the function on \(\mathcal{P}_n\) that is +1 (resp. -1) on even (resp. odd) permutations. Define

\[e_1 \otimes \cdots \otimes e_n \in \otimes^n \mathcal{H}\]

by

\[e_1 \otimes \cdots \otimes e_n = \frac{1}{(n!)^2} \sum_{\pi \in \mathcal{P}_n} \varepsilon(\pi) [e_{\pi(1)} \otimes \cdots \otimes e_{\pi(n)}]\]  \((178.2)\)

and define \(\Lambda^n(\mathcal{H})\) to be the subspace of \(\otimes^n \mathcal{H}\).
Spanned if $\phi_1, \ldots, \phi_n$. The $(n!)^{-\frac{1}{2}}$ normalization factor is chosen so that if $\phi_1, \ldots, \phi_n$ are orthonormal, then $\phi_1 \wedge \ldots \wedge \phi_n$ has norm one. More generally,

$$(179.1) \quad (\phi_1 \wedge \ldots \wedge \phi_n, \eta_1 \wedge \ldots \wedge \eta_n) = \det (\phi_i, \eta_j)_{i, j = 1}^n$$

Given $A \in \mathbb{L}(\mathbb{H})$, $\Gamma_n(A)$ leaves $\Lambda^n(\mathbb{H})$ invariant,

$$\Gamma_n(A) \phi_1 \wedge \ldots \wedge \phi_n = \sum_{\pi \in \mathcal{P}_n} \text{sgn}(\pi) A \phi_{\pi(1)} \wedge \ldots \wedge A \phi_{\pi(n)}$$

$$= A \phi_1 \wedge \ldots \wedge A \phi_n$$

and we denote its restriction to $\Lambda^n(\mathbb{H})$ by $\Lambda^n(A)$.

We have from (178.1)

$$(179.2) \quad \Lambda^n(AB) = \Lambda^n(A) \Lambda^n(B)$$

When $n = 0$, we define $\Lambda^n \mathbb{H}$ to be $I$ and

$$\Lambda^n(A) \text{ as } I : \mathbb{C} \to \mathbb{C}.$$

The connection between determinants of finite dimensional operators and $\Lambda^n(\cdot)$ is given by: