We now consider an important example of a Fredholm operator that is not of the form $1 + k$ with $k$ compact. Thus the Fredholm operators are a genuinely larger class of operators than those of the form $1 + k$, $k$ compact.

Examples of this type are intimately related to the so-called theory of Riemann–Hilbert problems.

Let $S^1 = \{ z : |z| = 1 \}$ denote the unit circle in $\mathbb{C}$.

For any $n \geq 1$, let $H^{(n)} = L^2(S^1, d\theta ; \mathbb{C}^n)$

\[ \{ f = (f_1, \ldots, f_n)^T : \int_{0}^{2\pi} \| f(e^{i\theta}) \|^2 d\theta < \infty \}, \]

and let

\[ L^{(n)} = \{ a = a_k e^{ik} : a_k \in \mathbb{C}^n \text{ and } \sum_{-\infty}^{\infty} \| a_k \|^2 < \infty \}. \]

The Fourier transform $\mathcal{F}$ maps $H^{(n)}$ unitarily onto $L^{(n)}$

\[ \mathcal{F} x = \mathcal{F} f = \hat{f} = \sum_{k=-\infty}^{\infty} f_k e^{-ik\theta} \]

\[ \| f \|_{H^{(n)}}^2 = \sum_{k=-\infty}^{\infty} \| f_k \|_{\mathbb{C}^n}^2. \]
The inverse $\mathcal{F}^{-1}$ of $\mathcal{F}$ is given by the $L^2$ convergent series

\[(136.1) \quad \mathcal{F}^{-1} \mathcal{F} = \sum_{k=-\infty}^{\infty} \mathcal{F}_k e^{ik\theta} = \mathcal{F} \sum_{k=-\infty}^{\infty} \mathcal{F}_k e^{ik\theta} .\]

For $f \in H^{(n)}$, define the bounded complementary orthogonal projections in $H^{(n)}$

\[(136.4) \quad P_+ f = \sum_{k=0}^{\infty} \mathcal{F}_k e^{ik\theta}, \quad P_- f = \sum_{k=-\infty}^{-1} \mathcal{F}_k e^{ik\theta} .\]

\[(136.3) \quad \| P_+ \| = 1, \quad P_+ + P_- = 1_{H^{(n)}}, \quad P_+ P_- = 0, \quad P_+ \mathcal{F} = P_+ .\]

Let

\[(136.4) \quad K_+ = \text{ran} \ P_+ .\]

Clearly functions in $K_+$ have analytic continuations to $|z| < 14$ and $|z| > 14$ respectively, and if $g \in P_- f$, then

\[g(3) = O \left( 1/|z| \right) \] as $|z| \to \infty$.

Now let $h$ be a continuous function from $S'$ to $GL(n, \mathbb{C})$, the invertible $n \times n$ matrices with complex entries. Define the Toeplitz operator $T : K_+ \to K_+$.
\((37.0)\)
\[ T \phi = P_\lambda(h \phi) \quad \phi \in \mathbb{H}. \]

\((37.1)\)
Clearly \( T \in L(\mathbb{H}) \) and \( \|T\| = \|h\|_{\infty}. \)

We will prove the following result. Let \( w(h) \) be the winding number of \( \det(h(z)) \) on \( S' \), i.e.,
\[ w(h) = \frac{1}{2\pi i} \int_{S'} \log \det h. \]

\[ = \frac{1}{2\pi i} \left[ \log \det h(e^{i\theta}) - \log \det h(e^{i\phi}) \right] \]

\textbf{Proposition 37.2}

The operator \( T \) in \((37.1)\) is Fredholm and
\[(37.3)\]
\[ \text{ind } T = -w(h) \]

\textbf{Remark:} If \( w(h) \rightarrow 0 \), then clearly \( T \) cannot be of the form \( I + \) compact.

\textbf{Definition 37.4}

Two nowhere singular continuous \( n \times n \) matrix valued functions \( h_1(z) \) and \( h_2(z) \) on \( S' : \|z\| = 1 \) are homotopy equivalent if there exists a continuous map \( h(z,t) \) from \( S' \times [0,1] \) to \( GL(n, \mathbb{C}) \) with the property that
$h(3,0) = h_1(3)$ and $h(3,1) = h_2(3)$. We say that $h(3,t)$ provides a homotopy deformation of $h_1(3)$ to $h_2(3)$.

Lemma 138.1

Two nowhere singular, continuous $n \times n$ matrix valued functions on $S^1$ are homotopy equivalent if and only if $w(h_1) = w(h_2)$.

Proof: The winding number is clearly conserved under a homotopy deformation, so it remains to show that if $w(h_1) = w(h_2)$, then $h_1$ and $h_2$ are homotopy equivalent. It suffices to show that if $h(3)$ is a non-singular continuous $n \times n$ matrix valued function on $S^1$ with $w(h) = N$, then $h$ is homotopy equivalent to $\text{diag}(e^{iN\theta}, 1, \ldots, 1)$.

Step 1 By the Stone-Weierstrass Theorem, there exists an $n \times n$ matrix valued function $h^{(1)}(3)$ on $S^1$ such that
(139.1) $h^{(n)}(z)$ is a trigonometric polynomial

$$h^{(n)}(z) = \sum_{k=-N^{(n)}}^{N^{(n)}} a_k z^k$$

for some $N^{(n)} \leq \infty$ and suitable $n \times n$ matrices $a_k$.

(139.2) $h_t(z) = h(z) + t\left( h^{(n)}(z) - h(z) \right)$ is continuous and

and hence provides a homotopy from $h$ to $h^{(n)}$. We

nowhere singular for all $0 \leq t, \leq 1$.

We have $w(h^{(n)}) = w(h)$.

Step 2 For any $n \times n$ matrix $M = (M_{ij})$ let

$\Delta_1(M), \Delta_2(M), \ldots, \Delta_n(M)$ denote the leading sub-

determinants, $\Delta_1(M) = M_{11}, \Delta_2(M) = \det\left( M_{11} M_{22} \right),$ $\ldots,$

$\Delta_n(M) = \det M$.

Now

$$p(t_2) = \prod_{i=1}^{n} \Delta_i \left( h^{(n)}(s=1) + t_2 \right)$$

is a monic polynomial in $t_2$ of degree $\frac{n(n+1)}{2}$. Clearly

there exists $t_2^0 > 0$, arbitrarily small so that

(139.3) $p(t_2^0) \neq 0$, and

(139.4) $h^{(n)}(z) + t_2$ is nowhere singular on $S^1$ for all $0 \leq t_2 \leq t_2^0$. 

We have \( w(h^{(s)} + t_2^0) = w(h^{(s)}) = w(h) \).

**Step 3** The rational function

\[
d(\beta) = \prod_{j=1}^{n} \Delta; \( h^{(s)}(\beta) + t_2^0 \)
\]

is non-trivial as \( d(\beta = 1) = p(t_2^0) \neq 0 \), and hence

\( d(\beta) \) has at most a finite \# of zeros. It follows that there exists \( t_3^0 \geq 0 \), arbitrarily small, so that

\[
(140.1) \quad d(3 - t_3^0) \text{ has no zeros on } S^1, \text{ and}
\]

\[
(140.2) \quad h^{(s)}(3 - t_3^0) + t_2^0 \text{ is non-singular on } S^1 \text{ for all } 0 \leq t_3 \leq t_3^0.
\]

We have

\[
w(h^{(s)}) = w(h)
\]

where

\[
h^{(s)}(\beta) = h^{(s)}(3 - t_3^0) + t_2^0.
\]

**Step 4** As the leading sub-determinants \( \Delta_1(3), \ldots, \Delta_n(3) \) of \( h^{(s)}(3) \) are all non-zero on \( S^{(s)} \), it follows by Gaussian elimination that we may express

\[
(140.3) \quad h^{(s)}(\beta) = L(\beta) D(\beta) U(\beta)
\]
where \( L(3), D(3) \) and \( U(3) \) are continuous on \( S \), and

\[
L(3) = I + \hat{L}(3), \quad \text{where } \hat{L}(3) \text{ is strictly lower triangular}
\]

\[
U(3) = I + \hat{U}(3), \quad \text{where } \hat{U}(3) \text{ is strictly upper triangular, and}
\]

\[
D(3) = \text{diag} (D_1(3), D_2(3), \ldots, D_n(3)), \quad \text{where}
\]

\[
\begin{align*}
\Delta_1(3) &= \Delta_1(3) \\
\Delta_2(3) &= \Delta_1(3) \Delta_2(3) \\
& \vdots \\
\Delta_n(3) &= \Delta_1(3) \cdots \Delta_n(3)
\end{align*}
\]

Set

\[
h^{(3)}(3, t) = \left( I + (1-t) \hat{L}(3) \right) \Delta(3) \left( I + (1-t) \hat{U}(3) \right), \quad 0 \leq t \leq 1
\]

Then clearly \( h^{(3)}(3, t) \) gives a homotopy deformation from

\[
h^{(3)}(3, t = 0) = h^{(3)}(3) \quad \text{to} \quad h^{(3)}(3, t = 1) = D(3)
\]

In particular \( \omega(D) = \omega(h) \)

This fact is of course true, a fortiori, as \( \det(D(3)) = \det(h^{(3)}(3)) \).
Step 5. For $0 \leq t \leq 1$, consider

$$h^{(3)}(3, t) = \begin{pmatrix} D_1(3) & 0 \\ \vdots & \vdots \\ D_{n-2}(3) & 0 \\ 0 & \cos(t\pi)D_{n-1}(3) + \sin(t\pi)/D_{n}(3) \\ 0 & \sin(t\pi)/D_{n}(3) \end{pmatrix}$$

Then $h^{(3)}(3, t)$ gives a homotopy deformation from

$$h^{(3)}(3, 0) = D(3) \text{ to } h^{(3)}(3, 1) = \begin{pmatrix} D_1 & 0 \\ \vdots & \vdots \\ D_{n-2} & 0 \\ 0 & 0 \text{ (m.n, m.n) } \\ 0 & 0 \end{pmatrix}$$

But then

$$h^{(4)}(3, t) = \begin{pmatrix} D_1 & 0 \\ \vdots & \vdots \\ D_{n-2} & 0 \\ 0 & \text{(m.n, m.n) \text{ switch } D_{n-1} \text{ and } D_n \text{ with } D_{n-1} \text{ and } D_n} \\ 0 & \text{(m.n, m.n) \text{ switch } D_{n-1} \text{ and } D_n \text{ with } D_{n-1} \text{ and } D_n} \end{pmatrix}$$

gives a homotopy deformation from $h^{(4)}$ to

$$\text{diag}(D_1, \ldots, D_{n-1}, D_n, 1)$$

Continuing in this way, we see that $h$ is homotopy equivalent to $\text{diag}(D_1, D_2, \ldots, D_n, 1) \cdot \text{diag}(\det D, 1, \ldots, 1)$. 

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Step 6

Write \[ \det D(3) = e^{iN} D_0(3), \quad z = e^{i\theta} \]

where \( N = w(h) = w(D) \) and \( w(D_0) = 0 \). Set

\[ D(3,t) = e^{iN} e^{(1-t) \log D_0(3)} \]

As it follows that \( \log D_0(3) \) is continuous on \( S^1 \), and hence \( \text{diag} (D(3,t), 1, \ldots, 1) \) provides a homotopy

deformation from \( \text{diag} (D(3), 1, \ldots, 1) \) to \( \text{diag}(e^{iN}, 1, \ldots, 1) \)

Combining the above steps we obtain the desired homotopy

\( h(3,t), 0 \leq t \leq 1, \) taking \( h(3) \) to \( \text{diag}(e^{iN}, 1, \ldots, 1) \), proving Lemma 38.1.

Proof of Proposition 137.2

Define \( \mathcal{T}^\# : \mathbb{K}_+ \rightarrow \mathbb{K}_+ \) by

\[ \mathcal{T}^\# \varphi = P_z(h^{-1} \varphi) , \quad \varphi \in \mathbb{K}_+ \]

Then for \( \varphi \in \mathbb{K}_+ \)

\[ \mathcal{T}^\# T \varphi = P_z h^{-1} P_z h \varphi = \varphi - P_z h^{-1} P_z h \varphi \]
We show that
\[ k \mathcal{F} = - P_+ h^{-1} P_- h \mathcal{F} \]
is compact on \( \mathcal{H}^+ \).

For this purpose it is enough to show that
\[ P_+ h^{-1} P_- \]
is compact in \( \mathcal{H}^{(n)} \). Moreover as the compact operators are closed under norm convergence and finite linear combinations, it is sufficient by the Stone-Weierstrass theorem, to consider the case when
\[ h^{-1}(z) = z^k E \quad \text{for some} \ k \in \mathbb{Z} \quad \text{and some constant} \ E. \]

But then for
\[ f = \sum_{-\infty}^{\infty} f_j z^j \]
\[ P_+ h^{-1} P_- f = P_+ (z^k E \sum_{-\infty}^{\infty} f_j z^j) \]
\[ = \chi_k \left( \sum_{j=-\infty}^{\infty} f_j z^{j+k} \right) \]
where \( \chi_k = 0 \) if \( k \leq 0 \) and \( \chi_k = 1 \) if \( k > 1 \).

Thus \( P_+ h^{-1} P_- \) is finite rank and hence compact. In a similar way, we see that \( TT^* = 1 + \text{compact} \). Thus
\(T^+\) is a pseudo-inverse for \(T\) and hence \(T\) is Fredholm by Theorem 11.2.1.

Under the deformation \(h(\theta, t)\) in Lemma 13.8.1, taking \(h(t_0) = h(\theta)\) to

\[h(\theta, t) = \text{diag}(e^{iN\theta}, 1, \ldots, 1), \quad N = w(h),\]

\(h(\cdot, t)\) is continuous and invertible on \(S^1\) for all \(0 \leq t \leq 1\),

and hence \(T(h(\cdot, t))\), \(0 \leq t \leq 1\), is Fredholm by The preceding calculation. It follows then from Theorem 11.4.1 that

\[(14.5.1) \quad \text{ind } T = \text{ind } T_0\]

where \(T_0 = T(h_0), \quad h_0 = \text{diag}(e^{iN\theta}, 1, \ldots, 1)\)

Suppose \(N > 0\). Now if \(T_0 \varphi = 0, \quad \varphi = (\varphi_1, \ldots, \varphi_N)\)

\(e^{iN\theta}\), then \(P_+ e^{iN\theta} \varphi = e^{iN\theta} \varphi = 0\) and \(P_+ \varphi = \varphi = 0, \quad i \in N\). Hence \(\varphi = 0\) and so \(\ker T_0 = \{0\}\). On the
other hand, under the non-degenerate pairing

\[ \langle g, f \rangle = \int_{\mathbb{S}} g(x) f(x) \, dx, \quad f, g \in H^{(n)} \]

otherwise, any linear functional $l$ on $H^{(n)}$ can be expressed uniquely by an element $g = g_e \in H^{(n)}$ as

\[ l(f) = \langle g, f \rangle \]

we may identify $H^+$ with $H^-$. Indeed, if $l \in H^+$, then $l f = d^0 P_+ f$ is in $H^{(n)'}$, and no

\[ l f = \langle g, f \rangle \]

for some $g \in H^{(n)}$. But then for $f \in H^+$,

\[ l(f) = l f = \int_{\mathbb{S}} g^{-T} f \, dx \]

\[ = \int_{\mathbb{S}} ((P - q) x^T + (P + q) f^T) f \, dx \]

\[ = \int_{\mathbb{S}} g^{-T} f \, dx \]

as $\int_{\mathbb{S}} (P + q) f \, dx = 0$. As $g^-$ is clearly unique, we see that we may identify $H^+$ with $H^-$. Also, for $g \in H^+$, $\tilde{e} \in H^-$ and $f \in H^+$,

\[ \langle T_0 g, f \rangle = \langle g, T_0 f \rangle = \langle \tilde{e}, P^{-1} h_0 f \rangle = \langle g, h_0 f \rangle \]
\[
\begin{align*}
= \int (g)^T h_0 \cdot d_3 &= \int (h_0^T g)^T \cdot d_3 \\
&= \int (p_{-h_0} g)^T \cdot d_3 \\
\text{Thus for } g \in \mathcal{H}_- = \mathcal{H}_+^*, \\
(147.1) \quad T_0' g &= p_{-h_0} g
\end{align*}
\]

Now as $T_0'$ is Fredholm, \text{codim } T_0 = \text{dim ker } T_0'. \forall g$

\[T_0' g = 0, \quad g = (g_1, \ldots, g_n)^T \in \mathcal{H}_-, \quad \text{Then we must have}
\]

\[p_{-E_{i_0}} g_i = 0 \quad \text{and} \quad g_i = 0 \quad \text{for} \quad 2 \leq i \leq n.
\]

Writing $g_i = \sum_{j=-\infty}^1 x_j e^{i j \theta}$ we find $x_j = 0$ for $i + N < 0$

$i < -N$. Thus $T_0' g = 0$ if and only if

\[g = \sum_{j=-N}^{-1} x_j e^{i j \theta},
\]

with $x_{-1}, \ldots, x_{-N}$ arbitrary. Thus \text{codim } T_0 = \text{dim ker } T_0' = N$. It follows that for $N > 0$, \text{ind } T < \text{ind } T_0 = 0 - N = -N = -\omega(h)$.

On the other hand, suppose $N < 0$. Then for $T^*$ as above,
we have by (120.2) \( \text{ind } T = -\text{ind } T^* \) as \( w(h^{-1}) \)
\[-w(h) = -N > 0 \text{ we conclude that } \text{ind } T^* = -w(h^{-1}) \]
\[= w(h) \quad \text{Thus again } \text{ind } T = -w(h) \text{, which completes} \]

the proof of Proposition 137.2 \( \square \)

As the final topic in this course we want

to define the class \( B_1(\mathcal{H}) \) of trace class operators

in a separable Hilbert space \( \mathcal{H} \). For such operators

\( T \in B_1(\mathcal{H}) \), we can define

\[ \text{det}(1 + T) \]

in an unambiguous way. The operators \( B_1(\mathcal{H}) \) thus

identify the class of (infinite dimensional) operators

for which "all" of the notions of finite dimensional

matrix algebra go through.