We now consider so-called analytic Fredholm Theory

Lecture 10 Let D be an open connected set in $\mathbb{C}$ and let X be a Banach space. We say that a map F from D to $\mathcal{L}(X)$ is meromorphic if there exists a discrete set $S \subset D$ with the following properties:

(123.1) S has no accumulation points in D

(123.2) $z \mapsto F(z)$ is analytic in $D \setminus S$

(123.3) At each point $s \in S$, $F(z)$ has a pole i.e. the Laurent expansion of $F(z)$ around s has only a finite number of negative powers

$$F(z) = \sum_{k=-n}^{\infty} F_k \, (z-s)^k$$

where $1 \le n = n(F, s) < \infty$.

Theorem 123.4 (Analytic Fredholm Theorem I).

Suppose X is a Banach space and let D be an open, connected subset of $\mathbb{C}$. Let $f : D \to \mathcal{L}(X)$ be an analytic operator-valued function such that $f(z)$ is compact for each $z \in D$. Then either

(124.1) $(1 - f(z))^{-1}$ exists for no $z \in D$, or

(124.2) $(1 - f(z))^{-1}$ exists as a meromorphic function on D i.e. for some discrete set $S \subset D$ with no accumulation points in D, $(1 - f(z))^{-1}$ is analytic in $D \setminus S$ and has at worst poles at points of S.

Furthermore, in case (124.2), the residues of $(1 - f(z))^{-1}$ at points of S are finite rank operators and if $s \in S$, then $\dim \ker (1 - f(s)) > 0$.

<u>Remark 124.3</u> In $X = \mathbb{C}^2$, $f(z) = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ is an example of an analytic operator on $D = \mathbb{C}$ for which (124.1) is true.

<u>Proof of Th^m 123.4</u> By a simple connectedness argument, it is enough to prove that any point $z_0 \in D$ has a neighborhood in which either (124.1) or (124.2) holds.

Let $z_0 \in D$. As $f(z_0) \in K(X)$, ind $(1 - f(z_0)) = 0$. Let $n = \dim \ker(1 - f(z_0)) = \operatorname{codim}(1 - f(z_0)) < \infty$, and suppose $\{x_1, \dots, x_n\}$ and $([y_1], \dots, [y_n])$ are bases for $\ker(1 - f(z_0))$ and $X / \operatorname{ran}(1 - f(z_0))$ respectively. By the Hahn-Banach Theorem, there exist $\{l_i\}_{i=1}^n \subset X'$ such that $l_i(x_j) = \delta_{ij}$, $1 \le i, j \le n$. Define $A \in \mathcal{L}(X)$ as follows:

(125.1) $$Ax = (1 - f(z_0))x + \sum_{i=1}^n l_i(x) y_i$$

Then A is a bijection from $X \to X$. Indeed, if $Ax = 0$ then $(1 - f(z_0))x = 0$ and $l_i(x) = 0$, $1 \le i \le n$, by the definition of the y_i's. But then $x = \sum_{j=1}^n \lambda_j x_j$ and

$\lambda_i = l_i(x) = 0$, $1 \le i \le n$. Thus $x = 0$, and so A is one-to-one. On the other hand, given $y \in X$, there exist ~~unique~~ $x, \alpha_1, \ldots, \alpha_n$ such that

$$y = (1 - f(z_0))x + \sum_{i=1}^{n} \alpha_i y_i .$$

Set

$$\tilde{x} = x + \sum_{i=1}^{n} \beta_i x_i \quad , \text{ where } \beta_i \equiv \alpha_i - l_i(x)$$

Then

$$A\tilde{x} = (1 - f(z_0))\tilde{x} + \sum_{j=1}^{n} l_j(\tilde{x}) y_j$$

$$= (1 - f(z_0))x + \sum_{j=1}^{n} \Big[l_j(x) + \sum_{i=1}^{n} \beta_i l_j(x_i) \Big] y_j$$

$$= (1 - f(z_0))x + \sum_{j=1}^{n} (l_j(x) + \beta_j) y_j$$

$$= (1 - f(z_0))x + \sum_{j=1}^{n} \alpha_j y_j$$

$$= y$$

Thus A is a bijection.

For z close to z_0, say $|z - z_0| < \varepsilon$, the operator

(126.1) $$A(z) \equiv 1 - f(z) + F \quad , \quad F(x) = \sum_{i=1}^{n} l_i(x) y_i \quad ,$$

$A(z_0) = A$, is clearly invertible. Hence

$$(1 - f(z)) = A(z) - F = (1 - G(z))A(z)$$

where $G(z) = F(A(z))^{-1}$ is an analytic finite rank

operator,

$$G(z)x = \sum_{i=1}^{n} l_i((A(z))^{-1}x)\, y_i .$$

Now as $G(z)$ is, in particular, compact, $(1 - G(z))^{-1}$ does

not exist if and only if

$$(1 - G(z))x = 0 \quad \text{for some} \quad 0 \neq x \in X$$

This is so if and only if for some α_i, $1 \le i \le n$, $x = \sum_{i=1}^{n} \alpha_i y_i$

$\neq 0$, and

(127.1) $$\sum_{i=1}^{n} \Big[\alpha_i - \sum_{j=1}^{n} l_i((A(z))^{-1}y_j)\,\alpha_j\Big] y_i = 0 ,$$

or equivalently

(127.2) $$d(z) \equiv \det(1 - \Lambda_A(z)) = 0$$

where

(127.3) $$\Lambda_A(z) = \big(l_i((A(z))^{-1}y_j)\big)_{1 \le i,j \le n}.$$

Thus $(1-G(z))^{-1}$ exists for no $|z-z_0|<\varepsilon$ if and only if $d(z)\equiv 0$ in $|z-z_0|<\varepsilon$. Alternatively, as $d(z)$ is analytic, if $(1-G(z'))^{-1}$ exists for some $z'\in\{|z-z_0|<\varepsilon\}$, then $d(z)$ vanishes at most at a discrete set $S_{z_0,\varepsilon}$ with no accumulation points in $\{|z-z_0|<\varepsilon\}$, and moreover $d(z)$ can only vanish to finite order at points of $S_{z_0,\varepsilon}$.

Now any $y\in X$ has a unique representation $y=r+\sum_{i=1}^{n}\beta_i y_i$ for $r\in \operatorname{ran}(1-F(z_0))$ and $\beta_i\in\mathbb{C}$. Write $r=Py$ and $\beta_i=\beta_i(y)=P_i y$, $1\le i\le n$. By the open mapping theorem $P, P_1,\dots,P_n$ are bounded linear maps on X. For $z'\in\{|z-z_0|<\varepsilon\}\setminus S_{z_0,\varepsilon}$, $(1-G(z'))^{-1}$ exists, and given $y=r+\sum_{i=1}^{n}\beta_i y_i$, we seek $x\in X$ such that $(1-G(z'))x=y$ in the form $x=r'+\sum_{i=1}^{n}\alpha_i y_i$, $r'\in\operatorname{ran}(1-F(z_0))$. We must have

$$r'+\sum_{i=1}^{n}\alpha_i y_i-\sum_{i=1}^{n} l_i\big((A(z'))^{-1}r'\big)y_i-\sum_{i=1}^{n}\Big(\sum_{j=1}^{n} l_i\big(A(z')^{-1}y_j\big)\alpha_j\Big)y_i$$

$$= r' + \sum_{i=1}^{n} \alpha_i y_i - \sum_{i=1}^{n} l_i\left((A(z'))^{-1} r'\right) y_i - \sum_{i=1}^{n}\left(\sum_{j=1}^{n} (\Lambda_A(z'))_{ij}\, \alpha_j\right) y_i$$

$$= r + \sum_{i=1}^{n} \beta_i y_i$$

and hence

(129.1) $$r' = r = Py$$

and if $\alpha = (\alpha_1, \dots, \alpha_n)^T$, $\beta = (\beta_1, \dots, \beta_n)^T$

and

$$l = \left(l_1((A(z))^{-1} r'), \dots, l_n((A(z))^{-1} r')\right)^T.$$

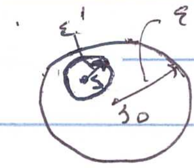
then

(129.2) $$\alpha = (1 - \Lambda_A(z'))^{-1}(\beta + l)$$

As $l_i((A(z'))^{-1} r) = l_i((A(z'))^{-1} Py)$, $1 \le i \le n$, is an analytic function in $\{|z - z_0| < \varepsilon\}$ and $\beta_i = P_i y$, it follows from the properties of $\alpha(z)$ that $x = Py + \sum \alpha_i y_i$ $= (1 - G(z))^{-1} y$ is a meromorphic function in $\{|z - z_0| < \varepsilon\}$, and hence $(1 - f(z)) = A(z)^{-1}(1 - G(z))^{-1}$ is meromorphic in $\{|z - z_0| < \varepsilon\}$. This proves that either (124.1) or (124.2) holds in D.

In case (124.2), let $s \in S \cap \{|z - z_0| < \varepsilon\}$ with z_0, ε as above, and let

$$(1-f(z))^{-1} = \sum_{k=-n}^{\infty} A_k (z-s)^k, \quad 1 \le n = n(s) < \infty,$$ be the Laurent series for $1-f(z)$ in the punctured disk $0 < |z-s| < \varepsilon$ around s. The residue term A_{-1} has the form



(130.1) $$A_{-1} = \oint_{|z-s|=\varepsilon'} (1-f(z))^{-1} \frac{dz}{2\pi i}$$

where $\oint$ denotes integrate over the (small) circle $\{|z-s|=\varepsilon'\} \subset \{|z-z_0|<\varepsilon\}$

Now if $y = r + \sum_{j=1}^{n} \beta_j y_j$, then $x = (1-f(z))^{-1} = r + \sum^{n} \alpha_i y_i$ and $(1-f(z))^{-1} y = A(z)^{-1} r + \sum_{j=1}^{n} \alpha_j A(z)^{-1} y_j$ where $\alpha_j = \alpha_j(z)$ are the meromorphic functions given in (129.2) above. Thus $A_{-1} y = \frac{1}{2\pi i} \sum_{j=1}^{n} \oint_{|z-s|=\varepsilon'} \alpha_j(z) A(z)^{-1} y_j \, dz$ from which it is clear that ran A_{-1} is spanned by the vectors

(130.2) $$\left\{ \frac{d^k}{dz^k} A(z)^{-1} y_i \Big|_{z=s}, \quad 0 \le k < k_0, \quad 1 \le i \le n \right\}$$

where k_0 is the degree of vanishing of $d(z) = \det(1 - \Lambda_A(z))$ at $z = s$. Thus the residues A_{-1} are finite rank operators.

Finally, as $f(s)$ is compact, $(1-f(s))^{-1}$ does not exist if and only if $\dim \ker(1-f(s)) > 0$. This completes the proof of the Theorem. □

<u>Theorem 131.1</u> (Analytic Fredholm Theorem II) Let X and Y be Banach spaces. Suppose $z \mapsto F(z)$ is an analytic map from an open, connected set $D \subset \mathbb{C}$ into the Fredholm operators from X to Y. Then either

(131.1) $\quad F(z)^{-1}$ exists for no $z \in D$, or

(131.2) $\quad F(z)^{-1}$ exists as a meromorphic function in D, i.e., for some discrete set $S \subset D$ with no accumulation points in D, $F(z)^{-1}$ exists and is analytic in $D \setminus S$ and has at worst poles at points of S.

Furthermore, in case (131.2), the residues of $F(z)^{-1}$ at points of S are finite rank operators, and if $s \in S$, then $\dim \ker F(s) > 0$

<u>Proof</u>: Let

$$B = \{ w \in D : \ F(z)^{-1} \text{ is meromorphic on an open neighborhood of } w \}$$

We prove that B is open and closed. By connectedness, this proves that either (131.1) or (131.2) holds.

The set B is open trivially and we only need to prove that B is closed. Suppose $z' \in D$. Then there exists $T(z') \in \mathcal{L}(Y, X)$ and $R_1(z') \in K(Y)$, $R_2(z') \in K(X)$ such that

(132.1) $$F(z')T(z') = 1 + R_1(z'), \quad T(z')F(z') = 1 + R_2(z').$$

For z near z',

$$F(z)\,T(z') = 1 + R_1(z') + \Delta_1(z) = \left[1 + R_1(z')(1+\Delta_1(z))^{-1}\right](1+\Delta_1(z))$$

where $\Delta_1(z) = (F(z) - F(z'))\,T(z')$ has small norm, and hence for z near z', there exist analytic functions $G_1(z)$ and $K_1(z)$ with $K_1(z)$ compact, such that

(132.2) $$F(z)\,G_1(z) = 1 + K_1(z)$$

Similarly, for z near z', there exist analytic functions $G_2(z)$ and $K_2(z)$ with $K_2(z)$ compact, such that

(132.3) $$G_2(z)\,F(z) = 1 + K_2(z).$$

Now suppose $\{z_n\} \subset B$ and $z_n \to z_\infty \in D$. We must show $z_\infty \in B$. By the above considerations, there exists a neighborhood N_∞ of $z' = z_\infty$ in which (132.2) and (132.3) hold. But as $z_n \to z_\infty$, there are points $\{\tilde{z}\}$ in N_∞ at which $F(\tilde{z})$ is invertible. Let $\tilde{z}$ be such a point. Then

$G_1(\tilde{z}) = (F(\tilde{z}))^{-1}(I + K_1(\tilde{z}))$ and hence $G_1(\tilde{z}) - (F(\tilde{z}))^{-1}$ is compact. Similarly $G_2(\tilde{z}) - F(\tilde{z})^{-1}$ is compact. For $z \in N_\infty$, set

$$\tilde{G}_1(z) = G_1(z) + F(\tilde{z})^{-1} - G_1(\tilde{z})$$

and

$$\tilde{G}_2(z) = G_2(z) + F(\tilde{z})^{-1} - G_2(\tilde{z})$$

Note that $\tilde{G}_1(\tilde{z}) = F(\tilde{z})^{-1} = \tilde{G}_2(\tilde{z})$. Clearly

$$F(z)\,\tilde{G}_1(z) = I + \tilde{K}_1(z)$$
$$\tilde{G}_2(z)\,F(z) = I + \tilde{K}_2(z)$$

where $\tilde{G}_i(z)$, $\tilde{K}_j(z)$ are analytic in N_∞ and $\tilde{K}_j(z)$ are compact. By construction, $\tilde{K}_1(\tilde{z}) = 0 = \tilde{K}_2(\tilde{z})$. It follows from the Analytic Fredholm Theorem 1, that $(1+\tilde{K}_1(z))^{-1}$ and $(1+\tilde{K}_2(z))^{-1}$ are meromorphic in N_∞. Thus

$$F(z)^{-1} = \tilde{G}_1(z)\,(I+\tilde{K}_1(z))^{-1} = (1+\tilde{K}_2(z))^{-1}\tilde{G}_2(z)$$

is meromorphic in N_∞. Take $N_{z_\infty} = N_\infty$. This proves that B is closed.

Furthermore, it follows from the above proof and the Analytic Fredholm Theorem I, that in the case (131.2), the residues of $F(z)^{-1}$ at points of S are finite rank. Also as $F(z)^{-1}$ exists at some point of D, it follows from the continuity of $F(z)$ and Theorem 114.1, that $\operatorname{ind}(F(z))=0$ for all $z\in D$. Thus $\dim\ker(F(z)) = \operatorname{codim}(F(z))$ for all $z\in D$, and hence at a point $s\in S$ we must have $\dim\ker(F(s))>0$ (and $\operatorname{codim}(F(s))>0$). This completes the proof of the Theorem. □

<u>Remark 134.1</u> Note that if $\operatorname{ind}(F(z))\neq 0$ for some $z=z_0$, then case (131.1) must hold.