Notes on traffic flow

Stephen Childress

March 22, 2005

1 The modeling of automobile traffic

When one thinks of modeling automobile traffic, it is natural to reason from personal experience and to visualize the car and driver as a coupled system, the driver responding to the surrounding vehicles and operating the car to make it become a part of the flow of freeway and city traffic. Thus the traffic is not just a mechanical process but one in which human decisions are involved, decisions which we have all experienced and can understand.

In our analysis of traffic we shall however step back from this personal view to take a broader perspective. Think of a traffic helicopter pilot looking down on the NYC highway grid. Looking at four miles of highway, the pilot will see a line of cars moving with various speeds. On some stretches, the traffic may be light and fast, on other stretches heavy and slow. To this observer the individual vehicles are not as important as the sense of overall *flow* of the cars. The reason why the cars in the lighter traffic move faster is clear to any driver, but to the observer in the helicopter it seems to be a property of the spacing of the cars. The closer they are together, the slower they move. Models of traffic flow try to exploit these observations and use them to formulate a set of assumptions which produce models which can be used to try to understand the peculiar and often frustrating occurences of daily driving. Have you ever experienced an hour of creeping traffic on a freeway, only to realize upon passing a patch of water on the road that this puddle is the cause of the problem, with every driver slowing down as they pass it? What is the effect of the closing of one lane one a four-lane throughway going to do to the traffic at rush hour? (We all know too well.) One ought to be able to understand some of the large effects of seemingly small causes.

In the picture just suggested, the cars are viewed in the large, almost as a moving gas or liquid. This kind of picture we will call a *continuum model of traffic flow*. We shall spend much of our time working from such a point of view. There is however another body of traffic theory based upon the point of view of the individual driver responding to surrounding traffic- just the way we would naturally want to think about driving. This kind of study is called *car-following theory*. We shall also look at some examples of this approach, which can be thought of as analogous to studying a gas by analyzing the motion of the individual molecules.

2 Formulation

Ultimately the traffic engineer is interested in how fast cars move through the traffic grid. Every car has a speedometer, and we all want to know how long it will take to go from A to B. Certainly one of the main quantitative measures of traffic is the speed of the cars in the traffic. Consider, for the sake of argument, a one-lane highway with cars in a line moving in the same direction. Since there is no passing, and cars cannot move through each other, the order of the cars is preserved, although they can move at slightly different speeds. Let the velocity of car "i'' be u_i. If the x-axis coincides with the road and the position of this car is $x_i(t)$ at time t, then the calculus tells us how u_i and x_i are related: u_i is the derivative with respect to time of x_i. Thus

$$u_i = \frac{dx_i}{dt} \tag{1}$$

Any discussion of traffic on our single-lane road must deal with a collection of vehicles, with positions $x_i(t), i = 1, 2, \ldots, N$ and velocities $u_i = \frac{dx_i}{dt}, i = 1, 2, \ldots, N$. The continuum approach to traffic takes the view that this collection of discrete objects should be replaced by a "moving continuum", a kind of fluid of vehicles. Such a fluid has a velocity at every value of x and at every time, and so we may define a *velocity field* by a function $u(x, t)$. The idea is that the variation of $u(x, t)$ with x should be on a scale of length (say a hundred yards) which is large compared to the size of a typical vehicle. Thus the value of $u(x, t)$ at a certain time t^* and a certain position x^* on the road should be the velocity of cars on that particular part of the road at that time.

If we know the velocity field for our road, how do we find the movement of an individual car? First we must specify the car. One way to do that is to choose a particular time, say $t = t_0$, and a particular position on the road, say $x = x_0$, and identify a car as being at that spot at that time. If we then want to know where this car is located at times $t > t_0$, we must use our knowledge of the velocity field, which tells us how fast any car is going when at position x and time t. Thus if $x(t)$ is the position of our car, we know that $x(t_0) = x_0$ but also that

$$\frac{dx}{dt} = u(x(t), t). \tag{2}$$

This last equation is the crucial one, since it relates the overall velocity field to the function $x(t)$ for the particular car which was located at x_0 at time t_0. We sometime will call $x(t)$ the *Lagrangian coordinate* of the car. This name pays respect to the French mathematician Lagrange (1736-1813), who introduced the description of a fluid by tracking the motion of individuals particles of the fluid. Also, $u(x, t)$ is sometimes called the *Eulerian* velocity field, after the Swiss mathematician Euler (1707-1783), who introduced the description of fluid motion using a velocity field.

Note that the problem of locating the position of our car, summarized as

$$\frac{dx}{dt} = u(x(t), t), \quad x(t_0) = x_0, \tag{3}$$

where $u(x,t)$ is a given function, amounts to solving an ordinary differential equation of first order with an initial condition at the time t_0.

Problem 1: Consider the units of x to be in miles. On the stretch of road $0 < x < 4$ cars are accelerating from a red light, and the velocity field is found to be $u(x,t) = 10x + 30t$ miles per hour where $t > 0$ is measured in hours. What is the Lagrangian coordinate of the car which was located at $x = 1.5$ at time $t = 0$? To answer this we must solve

$$\frac{dx}{dt} = 10x + 30t, \quad x(0) = 1.5. \tag{4}$$

Using the integrating factor e^{-10t}, we have $\frac{d}{dt}(e^{-10t}x) = 30te^{-10t}$. Integrating by parts, we get $e^{-10t}x = -(.3 + 3t)e^{-10t} + C$ or $x = -(.3 + 3t) + Ce^{10t}$. From our initial condition $x(0) = 1.5 = -.3 + C$, or $C = 1.8$. Thus $x(t) = -(.3 + 3t) + 1.8e^{10t}$.

We shall make frequent use of the $x - t$ plane in describing traffic flow. In figure 1 we show the velocity field just discussed, represented by the family of vehicle trajectories, each curve being the path of a car. (Note: the text sometimes uses the very nonstandard $t - x$ diagram instead of the standard $x - t$ diagram we use throughout here. Watch for this in your redading.)

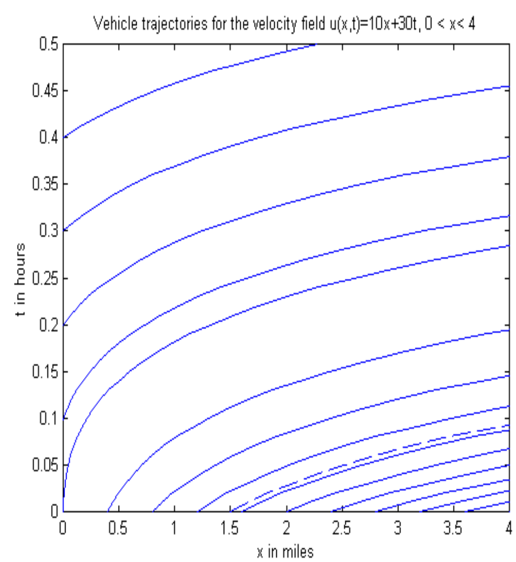

Figure 1. The vehicle trajectories for the velocity field $u(x,t) = 10x + 30t$, $0 < x < 4$. The dashed line is the path of the vehicle initially at $x = 1.5$ miles.

If a vehicle is traveling at constant velocity u, then in the $x - t$ plane its path is the straight line $x = u(t - t_0) + x_0$.

If we know the trajectories of all of the vehicles in a traffic flow, then we ought to be able to determine the velocity field of the flow. This is the problem inverse to the one we just solved, which was to find the car trajectory given the velocity field.

Problem 2 Let the cars' trajectories be given by $x = t^2 + 2tx_0 + x_0$. Note that $x(0) = x_0$, identifying the parameter x_0 as the initial position. Find the velocity field for this flow. To do this first compute the velocity, then use the two equations to eliminate x_0. Thus we have $\frac{dx}{dt} = u = 2t + 2x_0$, where the first equation tells us that $x_0 = \frac{x-t^2}{1+2t}$. Therefore $u(x,t) = 2t + \frac{2x-2t^2}{1+2t} = \frac{2t+2t^2+2x}{1+2t}$.

We point out that the parameter which determines which car we pick need not be the initial position.

Problem 3 let a family of vehicle trajectories be given by $x = ae^t + a^2, a > 0$. Find the corresponding velocity field $u(x,t)$. We have $u = ae^t$ and $a = (\sqrt{e^{2t} + 4x} - e^t)/2$. Thus $u(x,t) = e^t(\sqrt{e^{2t} + 4x} - e^t)/2$.

Problem 4 (See text, problem 57.6, page 264). Since every car has constant acceleration β, and starts (at $t = 0$ with zero velocity, we have $u = \beta t = \frac{dx}{dt}$. Since $x(0) = \beta$, we obtain $x = \beta t^2/2 + \beta$. To get the velocity field we elminate β between $u = \beta t$ and $x = \beta t^2/2 + \beta$. This gives $u(x,t) = \frac{xt}{1+t^2/2}$.

3 Traffic density

The second basic measure of traffic in a continuum model, in addition to the velocity field, is the *traffic density*. We again imagine a one-lane road with cars spread along it. The traffic density on this road associated with a given position x and time t, is the average number of vehicles per unit legnth of road at the position and time specified. Clearly to measure a density we need a stretch of road with enough cars on it to allow a reasonable statistical average. At the same time, we want to talk about the spatial variation of traffic density along the road, so the length over which we average should not be too long either, or else we will be getting to the natural scale of variation of the density. (You should read section 58 of the text for further discussion of this issue.)

We will use the traditional symbol for fluid density, namely ρ, for the traffic density. Thus $\rho(x,t)$ is the average number of cars per unit length at the position x and time t.

If all vehicles have length L (or else L is a good average length, and the spacing (or average spacing) between the cars is d, then each vehicle takes up $L + d$ units of road, so that approximates $\frac{1}{L+d}$ vehicles will be present per unit length of road. Thus the constant density of the traffic in this case is $\rho = \frac{1}{L+d}$.

4 Traffic flux

Again we think of our one-lane road, now having traffic with a certain density and velocity field. Another thing we need to think about is the common usage of the term *traffic flow*.We mean by this the rate at which What this is refering to is the rate at which cars an observer on the edge of the road, i.e. the *number of cars per unit time which cross a given point on the road*. We have seen the line accross a road that counts passing vehicles. This is being used to determine the traffic flow. Actually we shall prefer to use another term: *traffic flux*. The flux of cars is the wame as the flow (or better, the flow rate) of cars–i.e. the number of vehicles going by per unit time.

A key equation: Flux equals velocity times density If there are 100 cars per mile on a road, and each car is going 60 mph, then in one hour 60 miles worth of cars will pass an observer at the side of the foad, or $60 \times 100 = 6000$ cars per hour. This is the flux in this example., with $u = 60$ miles per hour and $\rho = 100$ vehicles per mile. The flux is $\rho u = 60 \times 100$ vehicles per hours (in the units, the "miles" cancel out).

We shall use the symbol q for flux. The flux is another key function which in general will depend, like u and ρ, upon x and t. thus

$$q(x,t) = \rho(x,t)u(x,t), \tag{5}$$

Problem 5 Assume that cars cannot leave or enter a one-way, two lane thoroughfare. At some point one lane is closed. After some confusion, the situation settles down to a steady state. The conditions well before the lane closing are seen to be constant and uniform; both lanes have the same density ρ_1 and the same velocity u_1. Note that ρ_1 is the density in each lane. Well after the closing of one lane, conditions are again uniform on the single lane with $u = u_2$ and $\rho = \rho_2$. What relation must exist between ρ_1, ρ_2, u_1, u_2? Since there are two lanes before the closing, the flux of cars on the throughfare there is $q_1 = 2\rho_1 u_1$. Well downstream of the closing the flux is $q_2 = \rho_2 u_2$. Since everthing is steady, and cars cannot leave or enter, the flux into the closing region must equal to that out of the region. Thus we have $2\rho_1 u_1 = \rho_2 u_2$.

Thus if the speed stays the same ($u_2 = u_1$), the density must double. In fact, higher density generally forces drivers to go slower, which drives the density up even further in order to preserve flux, leading to extreme slowdowns. To really exmine this problem, however, we need to study how flux and density are related through "conservation of vehicles" in roads without entrance or exit.

5 Conservation of the number of vehicles

Again we consider our 'bare" one-lane road (no entrances or exits). If we select some stretch of the road, between points $x = A$ and $x = B > A$ say, we know that the number of cars found to lie between A and B at some time t will in general depend upon the time t. If more cars flow into the segment AB than flow out of it, the number of vehicles within the segment will increase, and similar if more flow out than in, it will decrease. We can express this mathematically in terms of the *flux* at A and B. Namely, the rate of change of the number of vehicles in the segment, with respect to time, should equal the difference in flow rate or flux. If $N_{AB}(t)$ is this number of vehicles, then

$$\frac{dN_{AB}}{dt} = -q(B,t) + q(A,t). \tag{6}$$

On the other hand we know that N_{AB} can be comuted from the density by integration:

$$N_{AB}(t) = \int_A^B \rho(x,t)dx, \tag{7}$$

Thus we can rewrite our relation as

$$\frac{d}{dt}\int_A^B \rho(x,t)dx = -q(B,t) + q(A,t). \tag{8}$$

We say that this last result is a *global conservation law* for the vehicles on the road. Note that the signs on the right are consistent. If $q(B,t) > q(A,t)$ the more calls flow out than in, so N_{AB} will decrease in time.

For example, in problem 5 we considered the effect of a lane closing, and assume that the road had "settled into a steady state"; we then deduced that the fluxes were the same. We can see from our conservation law (8) that $q_(B,t) - q(A,t)$ whenever ρ becomes independent of time. This is because

$$\frac{d}{dt}\int_A^B \rho(x,t)dx = \int_A^B \frac{\partial \rho}{\partial t}dx, \tag{9}$$

since A, B are constants. Note the use of the partials on the right, since ρ also depends upon x.

The global conservation law implies a *local* conservation law, expressing the conservation of vehicle number on any stretch of road sufficiently long to allow us to assign a meaningful velocity and density function. To get the local relation we use the fundamental theorem of the calculus:

$$\int_A^B \frac{\partial q(x,t)}{\partial x}dx = q(B,t) - q(A,t). \tag{10}$$

Note that we have used a partial derivative with respect to x here, since q depends on both x and t. Otherwise this is the calculus I theorem.

Using (8),(9), and (10), we have

$$\int_A^B [\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x}]dx = 0. \tag{11}$$

Now this relation holds over an interval AB. We now use the following result (which we do not prove here): Let $f(x)$ be continuous on some closed interval $[\alpha, \beta]$ and assume that

$$\int_A^B f dx = 0 \tag{12}$$

for any interval $AB \in (\alpha, \beta)$. Then $f = 0$ for $\alpha < x < \beta$.

Using this result, we see that (11) implies that

$$\frac{\partial \rho(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0. \tag{13}$$

Using (5) we can write (13) as

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0. \tag{14}$$

Problem 6 let $x_A(t)$ be the position of one car on our one-lane road, and $x_B(t)$ be the position of another car many miles ahead. Then the number of cars between cars A and B is a constant, independent of time.

Of course this is obvious physically, but we can show it using calculus:

$$\frac{d}{dt}\int_{x_A(t)}^{x_B(t)} \rho(x,t)dx = \int_{x_A(t)}^{x_B(t)} \frac{d\rho(x,t)}{dt}dx + \rho(x_B(t),t)\frac{dx_B(t)}{dt} - \rho(x_A(t),t)\frac{dx_A(t)}{dt}. \tag{15}$$

But the right-hand-side of the last equation is just

$$\int_{x_A(t)}^{x_B(t)} \frac{d\rho(x,t)}{dt}dx + q(x_B(t),t) - q(x_A(t),t) = 0 \tag{16}$$

by (8).

6 Velocity as a function of density

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The equation we now have for vehicle conservation,

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} = 0, \tag{17}$$

is one relation involving two unknowns. Conventionally, we would need another relation to close the system in two unknowns. A major assumption that is often made by traffic modelers is that *velocity may be reasonably assumed to be a function of the density alone*. That is, we can assume $u = u(\rho)$ and our equation becomes a relation in ρ and its derivatives:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u(\rho))}{\partial x} = 0, \tag{18}$$

Such an equation is called a *partial differential equation (PDE) of first order*. It is a PDE because of the two variables involved, x, t, and the partial differentiations with respect to these variables. It is a first-order equation because only first partials are involved. To see this set $F(\rho) = \rho u(\rho)$. Then the equation (18) can be written

$$\frac{\partial \rho}{\partial t} + F'(\rho)\frac{\partial \rho}{\partial x} = 0. \tag{19}$$

We will focus later on how to solve this equation once $u(\rho)$ is given. For the moment our concern is whether or not this assumption is justified, and then what the function $u(\rho)$ should be.

On a single-lane open road this assumption seems to be fairly reasonable. An isolated car tends to have a maximum velocity of travel, either the result of speed limits or road conditions or driver caution, call it u_{max} Then for our function $u(\rho)$ we should take $u(0) = u_{max}$. We know that traffic speeds tend to go down with increasing traffic density, so we should assume that $du/d\rho < 0, \rho > 0$. Also there is surely a density, bumper to bumper traffic say, where the speed is essentially zero. Call this density ρ_{max}. If L is average car length, we could take $\rho_{max} = 1/L$. One widely used relation is

$$u(\rho) = u_{max}(1 - \rho/\rho_{max}). \tag{20}$$

Most of our discussion will concern the model of traffic flow which results from using (20).

Another argument might go as follows: assume that a deceleration rate A is tolerable to a driver, and therefore the driver will tend to stay a distance d behind a car so that if the car suddenly slowed to half (say) of its speed, the driver would be able to avoid a collision. The time it takes to decelerate from speed u to speed $u/2$ is $T = \frac{u}{2A}$, and the shortening of the distance between the cars to zero in this time implies that

$$\int_0^T (u - At)dt = d + uT/2, \tag{21}$$

giving $d = u^2/(8A)$. If the average car length is L, then $1/(d + L)$ should be the density when moving at speed u. Thus

$$\rho = \frac{1}{L + u^2/(8A)}, \tag{22}$$

, or

$$u = K\sqrt{\frac{1}{L\rho} - 1}, K = \sqrt{8AL}. \tag{23}$$

Since this relation goes to $+\infty$ as $\rho \to 0$, we must cut it off when $u = u_{max}$. Thus

$$u = \begin{cases} u_{max}, & \text{if } 0 < \rho < \rho_{min}, \\ \sqrt{8AL}\sqrt{\frac{1}{L\rho} - 1}, & \text{if } \rho > \rho_{min}. \end{cases} \tag{24}$$

Here

$$\rho_{min} = L^{-1}[1 + \frac{u_{max}^2}{8AL}]^{-1}. \tag{25}$$

In figure 2 we compare relations (20) amd (24) for some typical numbers.

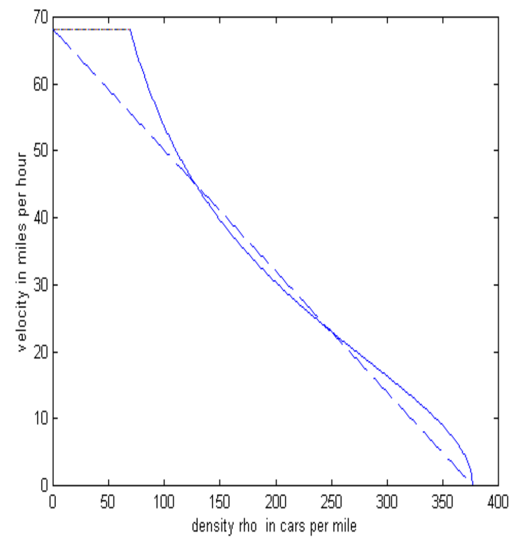

Figure 2. Velocity as a function of density using (20)(dotted line) and (24) (solid line). We take $L = 14feet$, $A = 20\ ft/sec^2$, $u_{max} = 100\ ft/sec$.

7 Car-following theory

We insert here a few ideas from car-following theory in order to contrast them with the continuum model we shall be looking at in more detail. There are any number of ways that modelers have tried to capture the driver response to surrounding traffic. The simplest assumes a given vehicle responds only to the car immediately in front of it (again restricting ourselves to the case of a single lane with no passing). One useful approach is to assume that car n responds to the car in front of it, car $n+1$ say, according to the difference of their two velocities. Let a fraction λ of the *velocity difference* of the two cars be eliminated by acceleration (or deceleration) of car n. Clearly deceleration will apply if $u_n > u_{n+1}$. If a_n is acceleration, then we should have

$$a_n = -\lambda(u_n - u_{n+1}), \tag{26}$$

In terms of car positions,

$$\frac{d^2x_n}{dt^2}(t) = -\lambda\big(\frac{dx_n}{dt}(t) - \frac{dx_{n+1}}{dt}(t)\big). \tag{27}$$

A somewhat more accurate model is to take into account a time delay T of the response of the driver in car n:

$$\frac{d^2x_n}{dt^2}(t+T) = -\lambda\big(\frac{dx_n}{dt}(t) - \frac{dx_{n+1}}{dt}(t)\big). \tag{28}$$

If all cars mover at the same speed u and are equally space a distance d apart, so that $d+L$ is the front to front distance between cars (L=car length), then one integration of (27) gives, since $1/(L+d)$ is then the uniform car density,

$$u = -\lambda(x_n - x_{n+1}) + C = \lambda(L+d) + C = \frac{\lambda}{\rho} - \frac{\lambda}{\rho_{max}}. \tag{29}$$

Here we have chosen the constant of integration to make $u = 0$ at $\rho = rho_{max}$. This gives us a velocity-density relation from a car-following theory. Since it goes to infinity as $\rho \to 0$ we need to again cut this off and take

$$u(\rho) = \begin{cases} u_{max}, & \text{for } 0 < \rho < \rho_{min}, \\ \lambda(\frac{1}{\rho} - \frac{1}{\rho_{max}}), & \text{for } \rho_{min} < \rho > \rho_{max}. \end{cases} \tag{30}$$

Here ρ_{min} is defined in terms of $u_{max}, \rho_{max}, \lambda$ by $u_{max} = \lambda(\frac{1}{\rho_{min}} - \frac{1}{\rho_{max}})$.

Let's examine the likely value of λ. It is useful here to deal with the unite feet and seconds, since we are talking about interactions between cars on the scale of seconds. It would seem reasonable to assume that a driver would try to eliminate the velocity difference in about 5 seconds, or about 1/5 of the difference per unit time, making $\lambda = 1/5$. To see how this plays out in a driving situation, suppose that two cars, car n and $n+1$, are both moving at 100 ft/sec and at $t = 0$ are separated by 200 feet, with car n at $x = 0$. At this moment car $n+1$ begins a constant deceleration, so that $u_{n+1}(t) = 100 - 20t$, so it will come to a stop in five seconds. We shall neglect the reaction time of driver n (i.e. the delay T), so we use (28) with $\lambda = 1/5$:

$$\frac{d^2 x_n}{dt^2}(t) + \frac{1}{5}\frac{dx_n}{dt}(t) = \frac{1}{5}(100 - 20t). \tag{31}$$

The solution of this inhomogeneous first-order ODE has the form

$$x_n = At^2 + Bt + C + De^{-t/5}. \tag{32}$$

The conditions are that $x_n(0) = 0$ and $dx_n/dt(0) = 100$. We find (verify this!)

$$x_n(t) = 200t - 10t^2 + 500(e^{-t/5} - 1). \tag{33}$$

Also we see by an integration, using $x_{n+1} = 200$, that

$$x_{n+1} = 100t - 10t^2 + 200. \tag{34}$$

At $t = 5$ seconds car $n + 1$ has come to rest at $x_{n+1} = 450$ feet while we can show that car n is still moving and in fact will collide with car $n + 1$ shortly after 5 seconds, see figure 3. Whether this value of λ is realistic is a matter of discussion. Problem it should be somewhat larger. However this calculation illustrates one aspect of the model which is unrealistic, namely the assumption that there is a constant λ which describes driver response. Obviously it is one thing to react calmly to speed changes, but quite another if you see a collision is imminent. In the calculation of figure 3 surely driver n is going to hit the brakes harder and harder into the stop.

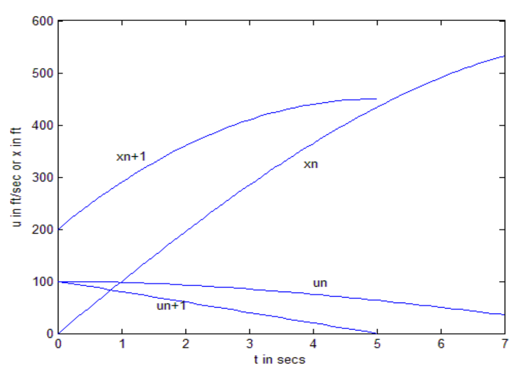

Figure 3. Car-following calculation of a collision.

We have dealt here only with the response of one car to a given motion of the car ahead. Clearly though if one considers N cars in a line, they will interact in a way which couples though a system of ODE's. Car-following theory studies these large systems of equations to get at the collective behavior. This is clearly computationally intensive and quite different from the continuum approach.

8 Linear traffic waves

Our PDE for the traffic density,

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u(\rho))}{\partial x} = 0, \tag{35}$$

will now be examined in more detail using the relation $u(\rho) = u_{max}(1-\rho/\rho_{max})$. This implies that $q(\rho) = \rho u(\rho) = u_{max}(\rho - \rho^2/\rho_{max})$. This is a simple quadratic function of ρ. We plot this in figure 4, using $\rho_{max} = 1/14$ vehicles per foot or $5280/14 \approx 377$ vehicles per mile , with $u_{max} = 70$ mph.

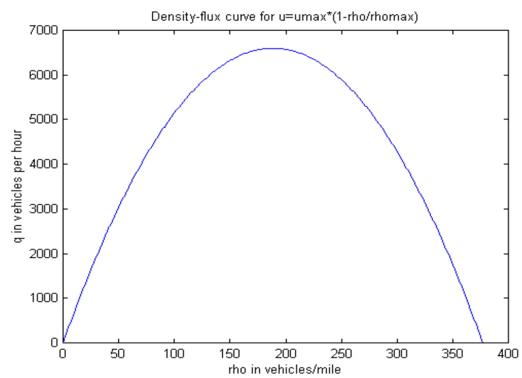

Figure 4. Flux as a function of density for when $u = u_{max}(1-\rho/\rho_{max})$. Here u_{max}= 70 mph and rho_{max} =377 vehicles per mile.

We can write (35) as

$$\frac{\partial \rho}{\partial t} + q'(\rho)\frac{\partial \rho}{\partial x} = 0, \tag{36}$$

This is an important form of the equation. Note that $q' = dq/d\rho$ has the dimensions of velocity. We are going to see that (36) expresses that waves, called *traffic waves* propagate with a velocity given by $q'(\rho)$. For the moment, however, we shall restrict attention to *linear traffic waves.*[1]. Let us suppose that $\rho = \rho_0 + \delta\rho$ in (36), where $\delta\rho \ll \rho_0$. That is, we want to consider s case where the traffic density is slightly perturbed from a constant density ρ_0. Note that if we put this into (36), we will need to expand q' in a Taylor series,

$$q'(\rho_0 + \delta\rho) = q'(\rho_0) + q''(\rho_0)\delta\rho + \dots, \tag{37}$$

but we see that the terms in δ can actually be dropped wince both partial derivatives in (36) are of order δ. Thus the linearized form of (36) is

$$\frac{\partial \rho}{\partial t} + q'(\rho_0)\frac{\partial \rho}{\partial x} = 0, \tag{38}$$

[1] "When in doubt, be wise......linearize."

Note that we could have written the partials as acting on $\delta\rho$. We prefer to include the constant ρ_0 to express this as an equation for the total density. The important point is that $q'(\rho_0)$, having the units of a velocity, is a *constant*, call it v_0. Now the equation

$$\frac{\partial\rho}{\partial t} + v_0\frac{\partial\rho}{\partial x} = 0, \tag{39}$$

has a very general solution of the form $\rho = f(x - v_0 t)$ for any differentiable function $f(x)$, as is easily seen by substitution in (39). To understand the mathematics behind this fact, it is helpful to about a general class of first order PDE's, which we will do in the next section. For the moment, we note simply that $\rho = f(x - v_0 t)$ describes a wave moving with velocity v_0. for $v_0 > 0$ the wave moves to the right, the opposite sign moving to the left. For example if $f(x) = \sin x$, so $\rho = \sin(x - v_0 t)/$ The point x, t such that $x - v_0 t = \pi/2$ is at the crest of a wave, and it moves in the $x - t$ plane along the straight line $x = v_0 t + \pi/2$. Thus the solutions of (39) represent *linear traffic waves* The velocity v_0 is given by

$$v_0 = u_{max}(1 - 2\rho_0/\rho_{max}). \tag{40}$$

It is important to realize that this velocity is *relative to the road surface*. Note that when $\rho_0 \approx 0$ we have $v_0 \approx u_{max}$. This is reasonable, since it says that the density changes are propagating with the velocity of the cars when there are few cars on the road. It also means that the traffic waves move with the traffic (again reasonable for light traffic). At the other extreme, when $\rho \approx \rho_{max}$, we see that $v_0 \approx -u_{max}$. Here of course cars are moving slowly, but the wave moves backwards relative to the car's motion at the high speed of u_{max}. An example of such a high-density wave is seen when cars are slowing moving in tightly packed traffic and a car suddenly stops. The wave of red brake lights can move toward a driver extremely quickly, the cause of many a rear-ender.

A helpful analysis and interpretation of this backward motion of the wave in terms of conservation of vehicles is given on page 311 of the text.

9 Solution of a class of first-order PDE's using characteristic curves

This will be a mathematical digression into solving first-order PDEs using the method of *characteristic curves*. Consider a function $f(x, t)$ satisfying a first-order linear PDE of the form

$$\frac{\partial f}{\partial t} + v(x, t)\frac{\partial f}{\partial x} = 0. \tag{41}$$

We shall view this equation as saying that f is not changing along a curve $x = x(t)$. Thus if

$$\frac{d}{dt}f(x(t), t) = 0, \tag{42}$$

we can use the chain rule to obtain

$$0 = \frac{\partial f}{\partial t} + \frac{dx}{dt}\frac{\partial f}{\partial x} = 0. \tag{43}$$

Comparing (41) and (43) we must have

$$\frac{dx}{dt} = v(x,t). \tag{44}$$

Since v is a given function of x,t, this is an ODE for $x(t)$. The general solution structure of such an ODE is as a set of integral curves $\phi(x,t) = constant$. On any such curve we see from (42) that f will also be constant. Since the curves of constant ϕ and constant f coincide, f must be a function of ϕ alone,

$$f(x,t) = F(\phi(x,t)). \tag{45}$$

The function F can be selected by supplying an appropriate condition. We will generally be interested in applying an initial condition, e.g. $f(x,0) = f_0(x)$. In that case f must be such that $f_0(x) = F(\phi(x,0))$. This last equation can in principle be solved for $x(\phi)$, and then $f(x,t) = f_0(x(\phi(x,t)))$.

We illustrate this method with two examples:

$$(1)\ \ \frac{\partial f}{\partial t} + te^{-x}\frac{\partial f}{\partial x} = 0, \quad f(x,0) = x. \tag{46}$$

Here we have

$$\frac{dx}{dt} = te^{-x}. \tag{47}$$

Integrating using separation of variables, we obtain

$$e^x - t^2/2 = \text{constant}. \tag{48}$$

Thus $f = F(e^x - t^2/2)$ is a solution for any differentiable function F. (Verify this by substitution.) Since $f(x,0) = x$, we see that $F = log$, so the solution is $f = log(e^x - t^2/2)$.

$$(2)\ \ \frac{\partial f}{\partial t} + \frac{1+x}{1+t}\frac{\partial f}{\partial x} = 0, \quad f(x,0) = \sin x. \tag{49}$$

From $dx/dt = (1+x)/(1+t)$ that $\phi = \frac{1+x}{1+t} =$ constant. Then $F(\phi) = \sin(\phi - 1)$ from the initial condition (note that we get this by solving $\phi = \frac{1+x}{1+t}$ with $t = 0$ for x as a function of ϕ, then put this function in $\sin x$), and the solution is

$$f(x,t) = \sin\big(\frac{x-t}{1+t}\big). \tag{50}$$

Again, this should be checked by differentiation.

The curves $\phi =$ constant along which f is constant are called *characteristic curves*. Although we have considered only a special class of first-order equations, the method of characteristic curves applies to first-order equations of general form. In two variables x, t, the general equations we can consider have the general nonlinear, inhomogeneous form

$$A(f,x,t)\frac{\partial f}{\partial t} + B(f,x,t)\frac{\partial f}{\partial x} = C(f,x,t) \tag{51}$$

We will consider the nonlinear case, but will restrict attention to our model of traffic flow.

10 Characteristic curves and the solution of the traffic flow equation

We now want to consider the solution

$$\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x} = 0, \quad v(\rho) = u_{max}(1 - 2\rho/\rho_{max}) \tag{52}$$

using the method of characteristics. Again we try to view this equations as constancy of $\rho(x(t), t)$ along a curve $x = x(t)$ in the $x - t$ plane. We then wee that we must have

$$\frac{dx}{dt} = v(\rho(x(t), t)). \tag{53}$$

This looks like a complicated problem for x, but notice that, since ρ *itself is a constant on the characteristic curve, so is* $v(\rho)$. Thus the RHS of the last equation is in fact a constant, independent of t. Thus we have

$$x = vt + x_0 \tag{54}$$

as the characteristic curve which starts at $x = x_0$ when $t = 0$. Since v is constant, this a straight line! We can moreover identify the constant value of v with the value at $(x, t) = (x_0, 0)$:

$$v = v(\rho(x_0, 0)). \tag{55}$$

We illustrate the situation by considering the following distribution of density at $t = 0$:

$$\rho(x,0) = \begin{cases} 200, & \text{for } x < 0, \\ 200(1 - x/2), & \text{for } 0 < x < 1, \\ 100, & \text{for } x > 1. \end{cases} \tag{56}$$

Thus we see that $v = 70(1 - 400/377) \approx -4.3$ mph for characteristics coming out of the negative x-axis, and equals $v = 70(1 - 200/377) \approx 33$ mph for those emerging from $x > 1$.

In between we have a gradual transition, $v = -4.3 + 37.3x$. We draw these characteristics in figure 5.

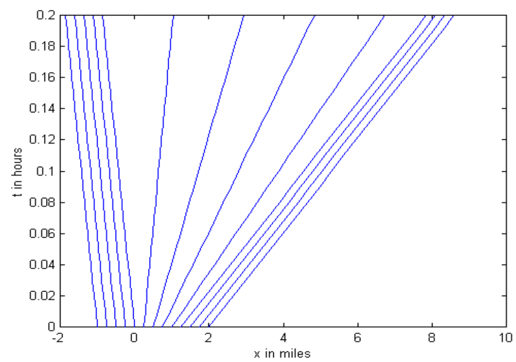

Figure 5. Solution of the traffic flow equation by the method of characteristics with initial density given by (56). A time interval of 12 minutes is shown.

Notice that, since ρ is constant on each characteristic, we have a way of computing the traffic density as a function of x for any time. Indeed we see that for $x < -4.3t$ we have $\rho = 200$. for $x > 1 + 33t$, we have $\rho = 100$. In between we can solve

$$x = (-4.3 + 37.3x_0)t + x_0 \tag{57}$$

for $x_0(x, t)$:

$$x_0 = \frac{x + 4.3t}{1 + 37.3t}. \tag{58}$$

In the region in between we have $\rho = 200(1 - x_0/2)$, and substituting from (58) we obtain

$$\rho(x, t) = 100\frac{2 + 70.3t - x}{1 + 37.3t}. \tag{59}$$

This $x - t$ diagram shows what happens to a gradually decreasing traffic density. The traffic waves are waves of constant density. The waves speeds carry the density outward. The overall pattern is of a "fan", and in fact the phenomenon is called an *expansion fan*.

We have here an example of solving the nonlinear traffic flow equation using the *method of characteristics*. Note that the main way it differs from solving for linear traffic waves is that in the linear case all of the characteristics have the same slope, since $v = v_0$ is the same for all characteristics. In the nonlinear case the characteristics are still straight lines, but their slope varies and is determined by the density at the point on the initial line where they emerge.

The method of characteristics applies to any equation of the form $\frac{\partial \rho}{\partial t} + v(\rho)\frac{\partial \rho}{\partial x} = 0$. We first express the equation for the characteristic lines in the form

$$x = v(\rho(x_0, 0))t + x_0. \quad (60)$$

We then try to solve (60) for x_0 as function of x, t. Since we know $\rho(x_0, 0) = \rho_0(x_0)$, $rho(x, t)$ can be obtained by inserting the function $x_0(x, t)$:

$$\rho(x, t) = \rho_0(x_0(x, t)). \quad (61)$$

To illustrate this, consider the problem

$$\frac{\partial \rho}{\partial t} + \sqrt{\rho}\frac{\partial \rho}{\partial x} = 0, \quad x > 0. \quad (62)$$

Let the initial condition by $\rho(x, 0) = x, x > 0$. The characteristic curves are given by

$$x = \sqrt{\rho(x, 0)}t + x_0 = \sqrt{x_0}t + x_0. \quad (63)$$

Solving for $\sqrt{x_0}$ and then squaring, we get

$$x_0(x, t) = \frac{1}{2}(t^2 - t\sqrt{t^2 + 4x}) + x. \quad (64)$$

The

$$\rho(x, t) = x_0(x, t) = \frac{1}{2}(t^2 - t\sqrt{t^2 + 4x}) + x. \quad (65)$$

That this is a solution can then be checked by partial differentiation.

In general we cannot do the inversion of $x = v(\rho_0(x_0))t + x_0$ to obtain $x_0(x, t)$. However the graphical construction of the solution in the x, t plane is still possible using the method of characteristics.

11 Traffic flow when a red light turns green

To simplify our equations we now assume that u is measured in units of u_{max} and that density is measured in units of ρ_{max}. This has the effect of making

$$q(\rho) = \rho(1 - \rho). \quad (66)$$

If x is regarded as in miles, the unit of time is them $1/u_{max}$ hours. It is therefore useful to make $u_{max} = 60$ so that time is measured in minutes. For example, speed 1 for 60 units of time (minutes) gives 60 miles.

We are interested in the situation when a red light turns green. If this occurs at $t = 0$, then at $t = 0$ the density is given by

$$\rho(x, 0) = \begin{cases} 1, & \text{for } x < 0, \\ 0 & \text{for } x > 0. \end{cases} \quad (67)$$

Now for characteristics emerging from $x < 0$ we see that $v(1) = 1 - 2 = -1$ and so the characteristics are $x = -t + x_0$ when $x_0 < 0$. Since $v(0) = 1 - 0 = 1$, the characteristics are $x = t + x_0$ when $x_0 > 0$, see figure 6.. The question is, what happens in the blank area not covered by these characteristics?

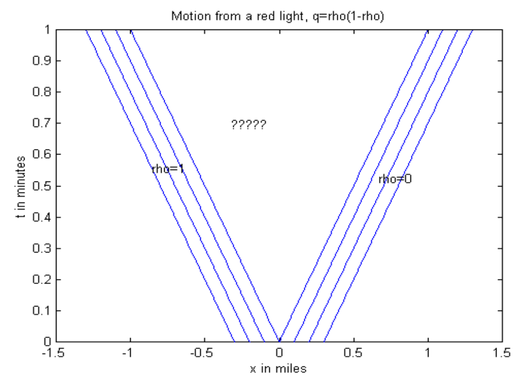

Figure 6. Characteristics for $q = \rho(1-\rho)$ when $\rho = 1, x < 0, = 0x > 0$.

To see what goes on we modify the problem so there is a continuous change of density instead of a discontinuity. Let ϵ be a small positive number. Let the initial density then be

$$\rho(x,0) = \begin{cases} 1, & \text{for } x < 0, \\ (1 - x/\epsilon), & \text{for } 0 < x < \epsilon, \\ 0, & \text{for } x > \epsilon. \end{cases} \tag{68}$$

Now in the transition region we have the characteristic curves given by

$$x = (1-2\rho(x_0,0))t + x_0 = (1-2(1-x_0/\epsilon))t + x_0 = -t + 2x_0 t/epsilon + x_0, \tag{69}$$

Solving for $x_0(x,t)$,

$$x_0 = \frac{x+t}{1+2t/\epsilon}. \tag{70}$$

The in the transition region we have

$$\rho(x,t) = \rho_0(x_0(x,t)) = 1 - \frac{x+t}{\epsilon+2t} = \frac{\epsilon+t-x}{\epsilon+2t}. \tag{71}$$

We can now let $\epsilon \to 0$ to obtain

$$\rho(x,t) = \frac{t-x}{2t}. \tag{72}$$

Note that these are straight lines coming out of the origin. The line $x = -t$ gives $\rho = 1$ as it should; the line $x = t$ gives $\rho = 0$ as it should. We show this *expansion fan* in figure 7

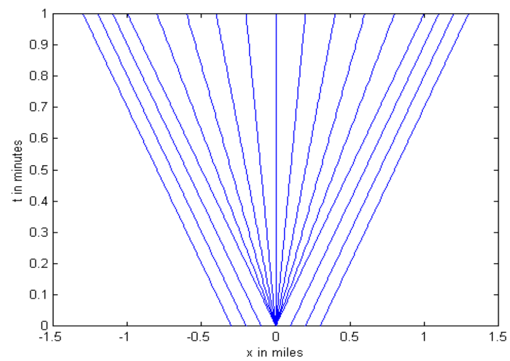

Figure 7. The expansion fan given by (72) fills in the gap of figure 6.

This function shows how the density varies smoothly from 1 to 0 as the cars accelerate. By supposing an initial, discontinuous density at $t = 0$, we have in effect modeled the turning on of a green light for traffic that was initially stopped at maximum density unity in $x < 0$.

We now show how the expansion fan can be computed without going through the exercise of using a continuous transition. We realize that the characteristics *must* emanate from (0,0) as straight lines, and so we must have $\rho = R(x/t)$ for some function R in the transition region. We now try to solve $\rho_t + (1 - 2\rho)\rho_x$ by a function $R(x/t)$. Letting $\eta = x/t$, the partials give

$$t^{-1}\frac{dR}{d\eta}(-\eta + (1 - 2R)) = 0, \tag{73}$$

or

$$R = \frac{1-\eta}{2} = \frac{t-x}{2t}. \tag{74}$$

We thus obtain the transition density directly.

11.1 Some properties of the traffic from a red light

In order to see better the numbers associated with motion from a red light, we want to restore units and let $q = \rho u_{max}(1 - \rho/\rho_{max})$. The first question we ask is, how long do you have to wait from the time the light turns green before you start to move? The second is, what is the path of your car once you begin to move? The third is, how close to the red light do you have to be to insure that you get thorugh trhe light in one cycle?

We now answer each of these questions in our model. Although the q versus ρ relation we are using is not the best in all situations, you will be able to see how to answer these questions given any $q(\rho)$, although the solution of the equations may be harder for other q.

Waiting time. The leftmost radial from the origin of the expansion fan is the traffic wave corresponding to ρ_{max}, having velocity $-u_{max}$. If the car in question is a distance D behind the light, the waiting time until the wave arrives at this position is thus D/u_{max}. In city traffic u_{max} might be 30 mph. Taking car spacing as 20 feet, we compute the waiting time per car As (20/5280)(1/30) hours or (20/5280)(1/30)(3600)= .45 seconds. Typical values are larger owing to human reaction time.

The vehicle path The calculation we do now is intended to emphasize the fact that the velocity of vehicles is completely independent of traffic wave velocity. Once a vehicle a distance D encounters the traffic wave which moves with velocity $-u_{max}$, the car will begin to move. The density in the expansion fan (problem 6 of homework 8) is

$$\rho(x,t) = \rho_{max}\big(\frac{u_{max}t - x}{2u_{max}t}\big). \tag{75}$$

Now the velocity of a car is $u = u_{max}(1 - \rho/\rho_{max})$. If we insert the ρ given by (75), we obtain the velocity of a car at position x, t of the plane. The path $x(t)$ of the car thus satisfies

$$\frac{dx}{dt} = u_{max}\Big(1 - \frac{1}{\rho_{max}}\big[\rho_{max}\big(\frac{u_{max}t - x}{2u_{max}t}\big)\big]\Big). \tag{76}$$

This simplifies to

$$\frac{dx}{dt} - \frac{x}{2t} = \frac{u_{max}}{2}. \tag{77}$$

Note that we get tghe RHS is we substitute $x = u_{max}t$ on the left. Thus we can write $x(t) = u_{max}t + X(t)$ and find the

$$\frac{dX}{dt} = \frac{X}{2t}. \tag{78}$$

Solving this equation by separation of variables, we get $X = C\sqrt{t}$ where C is an arbitrary constant. Thus the path of a car has the form

$$x = u_{max}t + C\sqrt{t}. \tag{79}$$

To find the constant C, we recall that the traffic wave with velocity $-u_{max}$ reaches the car a distance D behind the light at time $t_D = D/u_{max}$. Up to that time the car is stationary at $x = D$. Then the car begins to move. Thus we want to solve (79) with the condition

$$x(t_D) = -D, \text{ or } -D = D + C\sqrt{D/u_{max}}. \tag{80}$$

Thus $C = -2\sqrt{u_{max}D}$ and

$$x_{car}(t) = u_{max}t - 2\sqrt{u_{max}Dt}. \tag{81}$$

We show the car path as thge dotted line in figure 8 for the case where $D = .2$ miles. The time axis is in minutes if $u_{max} = 60mph$.

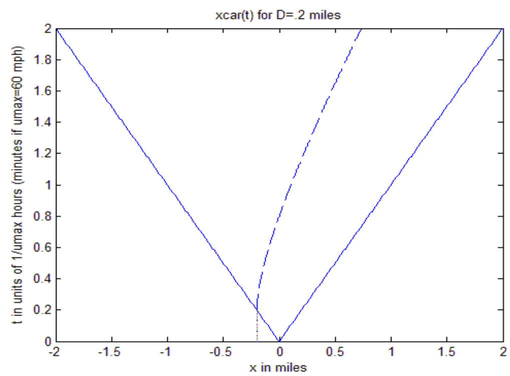

Figure 7. The path of the car starting a distance $D = .2$ miles behind the red light.

Which cars get through the light if the light is green for t_G time units? This is easy to see from (81). The last car to get through the light is the one starting from D_{last} where D_{last} makes $x_{car}(t_G) = 0$. Thus

$$0 = u_{max}t_G - 2\sqrt{u_{max}D_{last}t_G}, \tag{82}$$

or

$$D_{last} = u_{max}t_G/4. \tag{83}$$

In city driving, where $u_{max} = 30mph$ say, a light of 2 minutes or 1/30 hours will allow 1/4 mile of cars to move though.