

Notes on compressible flow

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1 Mechanical energy

We recall the two conservation laws from the first semester:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1)$$

Conservation of momentum:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \rho \frac{D u_i}{D t} = F_i + \frac{\partial \sigma_{ij}}{\partial x_j}, \quad \frac{D}{D t} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla. \quad (2)$$

Here σ_{ij} are the components of the viscous stress tensor:¹

$$\sigma_{ij} = -p \delta_{ij} + \mu \left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \delta_{ij} \nabla \cdot \mathbf{u} \right]. \quad (3)$$

We now want to use these results to compute the rate of change of total kinetic energy within a fixed fluid volume V . We see that

$$\begin{aligned} \frac{d}{dt} \int_V \frac{1}{2} \rho u^2 dV &= \int_V \left[\rho u_i \frac{\partial u_i}{\partial t} + \frac{1}{2} \frac{\partial \rho}{\partial t} u^2 \right] dV \\ &= \int_V \left(\left[-\rho u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_j} + F_i \right] u_i - \frac{1}{2} \frac{\partial \rho u_j}{\partial x_j} u^2 \right) dV \\ &= \int_V [u_i F_i + p \nabla \cdot \mathbf{u}] dV - \Phi + \int_S \left[-\frac{1}{2} \rho u_j u^2 + u_i \sigma_{ij} \right] n_j dS, \end{aligned} \quad (4)$$

where S is the boundary of V and Φ is the total viscous dissipation in V :

$$\Phi = \int_\phi dV, \quad \phi = \mu \left[\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 - \frac{2}{3} (\nabla \cdot \mathbf{u})^2 \right]. \quad (5)$$

¹You will find that Landau and Lifshitz allow a deviation from the Stokes relation. Recall that this relation makes the trace of the non-pressure part of the stress tensor (the deviatoric stress) zero. This deviation is accomplished by adding a term $\mu' \delta_{ij} \nabla \cdot \mathbf{u}$ to the stress tensor, where μ' is called the *second viscosity*. For simplicity, and since the exact form is not material to the problems we shall study, we neglect this second viscosity in the present notes. However for applications to real gases it should be retained as the Stokes relation does not hold for many gases.

We note from (4) that the contributions in order come from the work done by body forces, the work done by pressure in compression, viscous heating, flux of kinetic energy through S , and work done by stresses on S . We refer to (4) as the *mechanical* energy equation, since we have use only conservation of mass and momentum.

To put this expression into a different form we now complete the fluid equations by assuming a barotropic fluid, $p = p(\rho)$. Then

$$\int_V p \nabla \cdot \mathbf{u} dV = \int_S p \mathbf{u} \cdot \mathbf{n} dS - \int_V \mathbf{u} \cdot \nabla p dV. \quad (6)$$

But

$$\begin{aligned} \int_V \mathbf{u} \cdot \nabla p dV &= \int \rho \mathbf{u} \cdot \nabla \int \frac{1}{\rho} p'(\rho) d\rho dV \\ &= \int_S \rho \int \frac{1}{\rho} p'(\rho) d\rho \mathbf{u} \cdot \mathbf{n} dS + \int_V \frac{\partial \rho}{\partial t} \int \frac{1}{\rho} p'(\rho) d\rho dV. \end{aligned} \quad (7)$$

We now define $g(\rho)$ by

$$\int \frac{1}{\rho} p'(\rho) d\rho = g'(\rho), g(0) = 0. \quad (8)$$

and define e by

$$g = \rho e. \quad (9)$$

Noting that

$$\frac{d}{d\rho} \left(p - \rho \int \frac{1}{\rho} p'(\rho) d\rho \right) = - \int \frac{1}{\rho} p'(\rho) d\rho = -g' = -(\rho e)', \quad (10)$$

we have that

$$p - \rho \int \frac{1}{\rho} p'(\rho) d\rho = -\rho e. \quad (11)$$

Thus

$$\int_V \mathbf{u} \cdot \nabla p dV = - \int_S \rho e \mathbf{u} \cdot \mathbf{n} dS - \int \frac{\partial \rho}{\partial t} (\rho e)' dV. \quad (12)$$

Using this in (4) we obtain

$$\frac{d}{dt} \int_V E dV + \int_S E \mathbf{u} \cdot \mathbf{n} dS = -\Phi + \int_V u_i F_i dV + \int_S u_i \sigma_{ij} n_j dS, \quad (13)$$

where

$$E = \rho \left(e + \frac{1}{2} u^2 \right). \quad (14)$$

Note that if $\mu = 0$ and $F_i = 0$ then (13) reduces to a conservation law of the form

$$\frac{d}{dt} \int_V E dV + \int_S (E + p) \mathbf{u} \cdot \mathbf{n} dS \quad (15)$$

We note that E should have the meaning of energy, and we shall refer to e as the *internal energy* of the fluid (per unit mass). Then (13) can be viewed as an expression of the first law of thermodynamics $\Delta E = \Delta Q - W$, where ΔE is the change of energy of an isolated system (not flux of energy through the boundary), ΔQ is the heat *added* to the system, and W is the work done *by* the system.

The form of (13) can however be used as a model for formulating a more general energy equation, and we shall do this after first reviewing some of the basic concepts of reversible thermodynamics.

2 Elements of classical thermodynamics

Thermodynamics deals with transformations of energy within an isolated system. These transformations are determined by *thermodynamic variables*. These come in two types: *Extensive* variables are proportional to the amount of material involved. Examples are internal energy, entropy, heat. *Intensive* variables are not proportional to quantity. Examples are pressure, density, temperature.

We have just introduced a new scalar field, the *absolute temperature* T , and the *specific entropy* s . We shall also make use of *specific volume* v , defined by $v = 1/\rho$.

We now discuss the thermodynamics of gases. In general we shall assume the existence of an *equation of state* of the gas, connecting p, ρ, T . An important example is the equation of state of an *ideal* or *perfect* gas, defined by

$$pv = RT. \quad (16)$$

Here R is a constant associated with the particular gas. In general all thermodynamic variables are determined by ρ, p and T . With an equation of state, in principle we can regard any variable as a function of two independent variables.

We can now view our thermodynamic system as a small volume of gas which can do work by changing volume, can absorb and give off heat, and can change its internal energy. The first law then takes the differential form

$$dQ = de + pdv. \quad (17)$$

It is important to understand that we are considering here small changes which take place in such a way that irreversible dissipative processes are not present. For example, when a volume changes the gas has some velocity, and there could be resulting viscous dissipation. We are assuming that the operations are performed so that such effects are negligible. We then say that the system is *reversible*. If the changes are such that $dQ = 0$, we say that the system is *adiabatic*.

We define the following *specific heats* of the gas: The specific heat at constant pressure is defined by

$$c_p = \left. \frac{\partial Q}{\partial T} \right|_{dp=0} = \left(\frac{\partial e}{\partial T} \right)_p + p \left(\frac{\partial v}{\partial T} \right)_p. \quad (18)$$

Note that for an ideal gas $p\left(\frac{\partial v}{\partial T}\right)_p = R$.

The specific heat at constant volume is defined by

$$c_v = \left.\frac{\partial Q}{\partial T}\right|_{dv=0} = \left(\frac{\partial e}{\partial T}\right)_v. \quad (19)$$

We will make use of these presently.

The second law of thermodynamics for reversible systems establishes the existence of the thermodynamic variable s , the specific entropy, such that

$$dQ = Tds. \quad (20)$$

Thus we have the basic thermodynamic relation

$$Tds = de + pdv. \quad (21)$$

We now make use of (21) to establish an important property of an ideal gas, namely that its internal energy is a function of T alone. To see this, note from (21) that

$$\left(\frac{\partial e}{\partial s}\right)_v = T, \quad \left(\frac{\partial e}{\partial v}\right)_s = -p. \quad (22)$$

Thus

$$R\left(\frac{\partial e}{\partial s}\right)_v + v\left(\frac{\partial e}{\partial v}\right)_s = 0. \quad (23)$$

Thus e is a function of $s - R \ln v$ alone. Then, by the first of (22), $T = e'(s - R \ln v)$, implying $s - R \ln v$ is a function of T alone, and therefore e is also a function of T alone. Thus the derivative of e with respect to T at constant volume is the same as the derivative at constant pressure. By the definition of the specific heats, we have

$$c_p - c_v = R. \quad (24)$$

For an ideal gas it then follows that c_p and c_v differ by a constant. If both specific heats are constants, so that $e = c_v T$, it is customary to define that ratio

$$\gamma = c_p/c_v. \quad (25)$$

For air γ is about 1.4.

The case of constant specific heats gives rise to a useful model gas. Indeed we then have

$$ds = c_v \frac{dT}{T} + R \frac{dv}{v}. \quad (26)$$

Note that here the right-hand side explicitly verifies the existence of the differential ds . Using the equation of state of an ideal gas, the last equation may be integrated to obtain

$$p = k(s)\rho^\gamma, \quad k(s) = K e^{s/c_v}, \quad (27)$$

where K is a constant. The relation $p = k\rho^\gamma$ defines a *polytropic gas*.

3 The energy equation

The fundamental variables of compressible fluid mechanics of ideal gases are \mathbf{u}, ρ, p, T . We have three of momentum equations, one conservation of mass equations, and an equation of state. We need one more scalar equation to complete the system, and this will be an equation of conservation of energy. Guided by the mechanical energy equation, we are led to introduce the total energy per unit mass as $e + \frac{1}{2}u^2 = E/\rho$, and express energy conservation by the following relation:

$$\frac{d}{dt} \int_V E dV + \int_S E \mathbf{u} \cdot \mathbf{n} dS = \int_S u_i \sigma_{ij} n_j dS + \int_V F_i u_i dV + \int_S \lambda \nabla T \cdot \mathbf{n} dS. \quad (28)$$

We have on the right the working of body and surface forces and the heat flux to the system. The latter is based upon the assumption of Fick's law of heat condition, stating that heat flux is proportional to the gradient of temperature. We have introduced λ as the factor of proportionality. Given that heat flows from higher to lower temperature, λ as defined is a positive function, most often of ρ, T .

We now use (4) to eliminate some of the terms involving kinetic energy. Note the main idea here. Once we recognize that the energy of the fluid involves both kinetic and internal parts, we are prepared to write the first law as above. Then we make use of (4) to move to a more "thermodynamic" formulation. Proceeding we see easily that (28) becomes

$$\frac{d}{dt} \int_V \rho e dV + \int_S \rho e \mathbf{u} \cdot \mathbf{n} dS = \int_S \lambda \nabla T \cdot \mathbf{n} dS + \Phi - \int p \nabla \cdot \mathbf{u} dV. \quad (29)$$

This implies the local equation

$$\rho \frac{De}{Dt} - \nabla \cdot \lambda \nabla T - \phi + p \nabla \cdot \mathbf{u} = 0. \quad (30)$$

Using $T ds = de + pdv$ and the equation of conservation of mass, the last equation may be written

$$\rho T \frac{Ds}{Dt} = \nabla \cdot \lambda \nabla T + \phi. \quad (31)$$

This is immediately recognizable as having on the right precisely the heat inputs associated with changes of entropy.

There are other forms taken by the energy equation in addition to (30) and (31). These are easiest to derive using Maxwell's relations. To get each such relation we exhibit a principle function and from it obtain a differentiation identity, by using $T ds = de + pdv$ in a form exhibiting the principle function. For example if e is the principle function, then

$$\left(\frac{\partial e}{\partial s} \right)_v = T, \quad \left(\frac{\partial e}{\partial v} \right)_s = -p. \quad (32)$$

Then the Maxwell relation is obtained by cross differentiation:

$$\left(\frac{\partial T}{\partial v}\right)_s = -\left(\frac{\partial p}{\partial s}\right)_v. \quad (33)$$

We define the next principle function by $h = e + pv$, the *specific enthalpy*. Then $Tds = dh - vdp$. Thus

$$\left(\frac{\partial h}{\partial s}\right)_p = T, \quad \left(\frac{\partial h}{\partial p}\right)_s = v, \quad (34)$$

giving the relation

$$\left(\frac{\partial T}{\partial p}\right)_s = \left(\frac{\partial v}{\partial s}\right)_p. \quad (35)$$

The principle function and the corresponding Maxwell relation in the two remaining cases are:

The free energy $F = e - Ts$, yielding

$$\left(\frac{\partial p}{\partial T}\right)_v = \left(\frac{\partial s}{\partial v}\right)_T. \quad (36)$$

The free enthalpy $G = h - Ts$, yielding the relation

$$\left(\frac{\partial s}{\partial p}\right)_T = -\left(\frac{\partial v}{\partial T}\right)_p. \quad (37)$$

We illustrate the use of these relations by noting that

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial T}\right)_p dT + \left(\frac{\partial s}{\partial p}\right)_T dp \\ &= c_p \frac{dT}{T} - \left(\frac{\partial v}{\partial T}\right)_p dp, \end{aligned} \quad (38)$$

where we have used (37). Now for a perfect gas $\left(\frac{\partial v}{\partial T}\right)_p = R/p$, so that (31) may be written

$$\rho c_p \frac{DT}{Dt} - \frac{Dp}{Dt} = \nabla \cdot \lambda \nabla T + \phi. \quad (39)$$

In particular if c_p, λ are known functions of temperature say, then we have with the addition of (39) a closed system of six equations for \mathbf{u}, p, ρ, T .

4 Some basic relations for the non dissipative case $\mu = \lambda = 0$

In these case local conservation of energy may be written

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u}E) = \mathbf{u} \cdot \mathbf{F} - \nabla \cdot (\mathbf{u}p). \quad (40)$$

Using conservation of mass we have

$$\begin{aligned}\frac{D(e + \frac{1}{2}u^2)}{Dt} &= \frac{1}{\rho} \left(\mathbf{u} \cdot \mathbf{F} - \mathbf{u} \cdot \nabla p + \frac{p}{\rho} \frac{D\rho}{Dt} \right) \\ &= \frac{1}{\rho} \left(\mathbf{u} \cdot \mathbf{F} + \frac{\partial p}{\partial t} \right) - \frac{D}{Dt} \frac{p}{\rho}.\end{aligned}$$

Thus

$$\frac{D}{Dt} \left(e + \frac{1}{2}u^2 + \frac{p}{\rho} \right) = \frac{1}{\rho} \left(\mathbf{u} \cdot \mathbf{F} + \frac{\partial p}{\partial t} \right). \quad (41)$$

If now the flow is *steady*, and $\mathbf{F} = -\rho\nabla\Psi$, then we obtain a Bernoulli equation in the form

$$H \equiv e + \frac{1}{2}u^2 + \frac{p}{\rho} + \Psi = \text{constant} \quad (42)$$

on streamlines of the flow.

To see how H changes from streamline to streamline in steady flow, note that

$$dH = d\left(\frac{1}{2}u^2 + h + \Psi\right) = d\left(\frac{1}{2}u^2 + \Psi\right) + Tds + vdp, \quad (43)$$

so that we may write

$$\nabla H = \nabla\left(\frac{1}{2}u^2 + \Psi\right) + T\nabla s + \frac{1}{\rho}\nabla p. \quad (44)$$

But in steady flow with $\mu = 0$ we have $\rho\mathbf{u} \cdot \nabla\mathbf{u} + \nabla p = -\rho\nabla\Psi$, or

$$\nabla\left(\frac{1}{2}u^2 + \Psi\right) + \frac{1}{\rho}\nabla p = \mathbf{u} \times \boldsymbol{\omega}, \quad (45)$$

where $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is the vorticity vector. Using the last equation in (44) we obtain *Crocco's relation*:

$$\nabla H - T\nabla s = \mathbf{u} \times \boldsymbol{\omega}. \quad (46)$$

A flow in which $Ds/Dt = 0$ is called *isentropic*. From (31) we see that $\mu = \lambda = 0$ implies isentropic flow. If in addition s is constant throughout space, the flow is said to be *homotropic*. We see from (46) that in homotropic flow we have

$$\nabla H = \mathbf{u} \times \boldsymbol{\omega}. \quad (47)$$

Note also that in homotropic flow the Bernoulli relation (42) becomes (since $dh = vdp$)

$$\int \frac{c^2}{\rho} d\rho + \frac{1}{2}u^2 + \Psi = \text{constant} \quad (48)$$

on streamlines. here

$$c^2 = \left(\frac{\partial p}{\partial \rho} \right)_s \quad (49)$$

is the speed of sound in the gas.

4.1 Kelvin's theorem in a compressible medium

Following the calculation of the rate of change of circulation which we carried out in the incompressible case, consider the circulation integral over a material contour C :

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot \frac{\partial \mathbf{x}}{\partial \alpha} d\alpha, \quad (50)$$

where α is a Lagrangian parameter for the curve. then

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \oint_C \frac{D\mathbf{u}}{Dt} \cdot d\mathbf{x} + \oint_C \mathbf{u} \cdot d\mathbf{u}. \quad (51)$$

Using $D\mathbf{u}/Dt = -\nabla p/\rho - \nabla\Psi$, we get after disposing of perfect differentials,

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \int_S \frac{1}{\rho^2} (\nabla\rho \times \nabla p) \cdot \mathbf{n} dS. \quad (52)$$

Here S is any oriented surface spanning C . In a perfect gas, $Tds = c_v dT + pdv$, so that

$$T\nabla T \times \nabla s = \nabla \frac{p}{\rho R} \times p \nabla \frac{1}{\rho} = -\frac{T}{\rho^2} \nabla p \times \nabla \rho. \quad (53)$$

Thus

$$\frac{d}{dt} \oint_{C(t)} \mathbf{u} \cdot d\mathbf{x} = \int_S (\nabla T \times \nabla s) \cdot \mathbf{n} dS. \quad (54)$$

4.2 Examples

We now give a brief summary of two examples of systems of compressible flow equations of practical importance. We first consider the equations of *acoustics*. This is the theory of sound propagation. The disturbances of the air are so small that viscous and heat conduction effects may be neglected to first approximation, and the flow taken as homentropic. Since disturbances are small, we write $\rho = \rho_0 + \rho'$, $p = p_0 + p'$, $\mathbf{u} = \mathbf{u}'$ where subscript "0" denotes constant ambient conditions. If the ambient speed of sound is

$$\left(\frac{\partial p}{\partial \rho} \right)_s \Big|_0 = c_0^2, \quad (55)$$

we assume $\rho'/\rho_0, p'/p_0, |\mathbf{u}'|/c_0$ are all small. Also we see that $p' \approx c_0^2 \rho'$. With no body force, the mass and momentum equations give us

$$\frac{\partial \mathbf{u}'}{\partial t} + \frac{c_0^2}{\rho_0} \nabla \rho' = 0, \quad \frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{u}' = 0. \quad (56)$$

here we have neglected terms quadratic in primed quantities. Thus we obtain acoustics as a linearization of the compressible flow equations about a homogeneous ambient gas at rest.

Combining (56) we obtain

$$\left(\frac{\partial^2}{\partial t^2} - c_0^2 \nabla^2\right)(\rho', \mathbf{u}') = 0. \quad (57)$$

Thus we obtain the wave equation for the flow perturbations. If sound waves arise from still air, Kelvin's theorem guarantees that $\mathbf{u}' = \nabla\phi$, where ϕ will also satisfy the wave equation, with

$$p' = c_0^2 \rho' = -\frac{1}{\rho_0} \frac{\partial \phi}{\partial t}. \quad (58)$$

The second example of compressible flow is 2D steady isentropic flow of a polytropic gas with $\mu = \lambda = \Psi = 0$. Then

$$\mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = 0, \nabla \cdot (\rho \mathbf{u}) = 0. \quad (59)$$

Let $\mathbf{u} = (u, v)$, $q^2 = u^2 + v^2$. In this case the Bernoulli relation holds in the form

$$\frac{1}{2} q^2 + \int \frac{dp}{\rho} = \text{constant} \quad (60)$$

on streamlines. For a polytropic gas we have

$$\int \frac{dp}{\rho} = k \frac{\gamma}{\gamma - 1} \rho^{\gamma - 1} = \frac{1}{\gamma - 1} c^2. \quad (61)$$

Now we have, in component form

$$uu_x + vu_y + \frac{c^2}{\rho} \rho_x = 0, uv_x + vv_y + \frac{c^2}{\rho} \rho_y = 0, \quad (62)$$

and

$$u_x + v_y + u(\rho_x/\rho) + v(\rho_y/\rho) = 0 \quad (63)$$

Substituting for the *rho* terms using (62), we obtain

$$(c^2 - u^2)u_x + (c^2 - v^2)v_y - 2uv(u_x + v_y) = 0. \quad (64)$$

If now we assume irrotational flow, $v_x = u_y$, so that $(u, v) = (\phi_x, \phi_y)$, then we have the system

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} - 2\phi_x\phi_y\nabla^2\phi = 0, \quad (65)$$

$$\phi_x^2 + \phi_y^2 + \frac{2}{\gamma - 1} c^2 = \text{constant}. \quad (66)$$

5 The theory of sound

The study of acoustics is of interest as the fundamental problem of linearized gas dynamics. we have seen that the wave equation results. In the present section we drop the subscript “0” and write

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \nabla^2 \phi = 0, \quad (67)$$

where c is a constant phase speed of sound waves.

We first consider the one-dimensional case, and the initial-value problem on $-\infty < x < +\infty$. The natural initial conditions are for the gas velocity and the pressure or density, implying that both ϕ and ϕ_t should be supplied initially. Thus the problem is formulated as follows:

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \frac{\partial^2 \phi}{\partial x^2} = 0, \quad \phi(x, 0) = f(x), \quad \phi_t(x, 0) = g(x). \quad (68)$$

The general solution is easily seen to have the form

$$\phi = F(x - ct) + G(x + ct), \quad (69)$$

using the initial conditions to solve for F, G we obtain easily *D'Alembert's solution*:

$$\phi(x, t) = \frac{1}{2}[f(x - ct) + f(x + ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds. \quad (70)$$

In the (x, t) plane, a given point (x_0, t_0) in $t > 0$ is influenced only by the initial data on that interval of the x -axis lying between the points of intersection with the axis of the two lines $x - x_0 = \pm c(t - t_0)$. This interval is called the *domain of dependence* of (x_0, t_0) . Conversely a given point (x_0, t_0) in $t \geq 0$ can influence on the point with the wedge bounded by the two lines $x - x_0 = \pm c(t - t_0)$ with $t - t_0 \geq 0$. This wedge is called the *range of influence* of (x_0, t_0) . These lines are also known as the *characteristics* through the point (x_0, t_0) .

5.1 The fundamental solution in 3D

We first note that the three dimensions, under the condition of spherical symmetry, the wave equation has the form

$$\frac{\partial^2 \phi}{\partial t^2} - c^2 \left(\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} \right) = 0. \quad (71)$$

here $r^2 = x^2 + y^2 + z^2$. Note that we can rewrite this as

$$(r\phi)_{tt} - (r\phi)_{rr} = 0, \quad (72)$$

Thus we can reduce the 3D problem to the 1D problem if the symmetry holds.

Now we are interested in solving the 3D wave equation with a distribution as a forcing function, and with null initial conditions. In particular we seek the solution of

$$\phi_{tt} - c^2 \nabla^2 \phi = \delta(\mathbf{x})\delta(t), \quad (73)$$

with $\phi(\mathbf{x}, 0-) = \phi_t(\mathbf{x}, 0-) = 0$. Since the 3D delta function imposes no deviation from spherical symmetry, we assume this symmetry and solve the problem as a 1D problem. When $t > 0$ we see from the 1D problem that

$$\phi = \frac{1}{r} [F(t - r/c) + G(t + r/c)]. \quad (74)$$

(The change $r - ct$ to $t - r/c$ is immaterial but will be convenient here.) Now the term in G represents “incoming” signals propagating toward the origin from ∞ . Such waves are unphysical in the present case. Think of the delta function as a disturbance localized in space and time, like a firecracker set off at the origin and at $t = 0$. It should produce only outgoing signals. So we set $G = 0$. Also, near the origin $F(t - r/c) \approx F(t)$, so the $\delta(\mathbf{x})\delta(t)$ distribution would result, using $\nabla^2(1/r) = -4\pi\delta(\mathbf{x})$, provided

$$F(t - r/c) = \frac{1}{4\pi c^2} \delta(t - r/c). \quad (75)$$

Another way to do this is to integrate the left-hand side of (73) over the ball $r \leq \epsilon$, use the divergence theorem, and let $\epsilon \rightarrow 0$.

So we define the fundamental solution of the 3D wave equation by

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi c^2 r} \delta(t - r/c). \quad (76)$$

5.2 The bursting balloon problem

To illustrate solution in three dimensions consider the following initial conditions under radial symmetry. We assume that the pressure perturbation p satisfies

$$p(\mathbf{x}, 0) = \begin{cases} p_b, & \text{if } 0 < r < r_b, \\ 0, & \text{if } r > r_b. \end{cases} \equiv N(r) \quad (77)$$

Here p_b is a positive constant representing the initial pressure in the balloon. Now $p_t = c^2 \rho_t = -c^2 \rho_0 \nabla^2 \phi$. If the velocity of the gas is to be zero initially, as we must assume in the case of a fixed balloon, then

$$p_t(\mathbf{x}, 0) = 0. \quad (78)$$

Since rp satisfies the 1D wave equation, and P is presumably bounded at $r = 0$, we extend the solution to negative r by making rp an odd function. Then the initial value problem for rp is well defined in the D’Alembert sense and the solution is

$$rp = \frac{1}{2} [N(r - ct) + N(r + ct)]. \quad (79)$$

Note that we have both incoming and outgoing waves since the initial condition is over a finite domain. For large time, however, the incoming wave does not contribute and the pressure is a decaying “ N ” wave of width $2r_b$ centered at $r = ct$.

5.3 Kirchoff’s solution

We now take up the solution of the general initial value problem for the wave equation in 3D:

$$\phi_{tt} - c^2 \nabla^2 \phi = 0, \quad \phi(\mathbf{x}, 0) = f(\mathbf{x}), \quad \phi_t(\mathbf{x}, 0) = g(\mathbf{x}). \quad (80)$$

This can be accomplished from two ingenious steps. We first note that if ϕ solves the wave equation with the initial conditions $f = 0, g = h$, the ϕ_t solves the wave equation with $f = h, g = 0$. Indeed $\phi_{tt} = c^2 \nabla^2 \phi$ tends to zero as $t \rightarrow 0$ since this is a property of ϕ .

The second step is to note that the solution ϕ with $f = 0, g = h$ is given by

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi t c^2} \int_{S(\mathbf{x}, t)} h(\mathbf{x}') dS'. \quad (81)$$

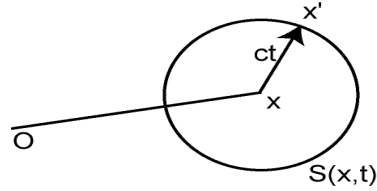


Figure 1. Definition of $S(\mathbf{x}, t)$ in the Kirchoff solution.

The meaning of S here is indicated in the figure. To verify that this is the solution, note first that we are integrating over a spherical surface of radius $4\pi c^2 t^2$. Given that h is bounded, division by t still leaves a factor t , so we obtain 0 in the limit as $t \rightarrow 0$. Also

$$\phi = \frac{t}{4\pi} \int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc) dS_{\mathbf{y}} \quad (82)$$

by a simple change of variable. Thus

$$\phi_t = \frac{1}{4\pi} \int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc) dS_{\mathbf{y}} + \frac{t}{4\pi} \int_{|\mathbf{y}|=1} c\mathbf{y} \cdot \nabla h(\mathbf{x} + \mathbf{y}tc) dS_{\mathbf{y}}. \quad (83)$$

The first term on the right clearly tends to $h(\mathbf{x})$ as $t \rightarrow 0$, while the second term tends to zero providing that h is a sufficiently well-behaved function.

We now show that (82) solves the wave equation. Using the divergence theorem we can write (83) in the form

$$\phi_t = \frac{1}{4\pi} \int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc) dS_y + \frac{1}{4\pi ct} \int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}') dV', \quad (84)$$

where $V(\mathbf{x}, t)$ denotes the sphere of radius ct centered at \mathbf{x} . Then

$$\phi_t = \frac{1}{4\pi} \int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc) dS_y + \frac{1}{4\pi ct} \int_0^{ct} \int_{S_{\rho(\mathbf{x})}} \nabla^2 h(\mathbf{y}) dS_y d\rho, \quad (85)$$

where $S_{\rho(\mathbf{x})}$ is the spherical surface of radius ρ centered at \mathbf{x} .

Now we can compute

$$\begin{aligned} \phi_{tt} &= \frac{c}{4\pi} \int_{|\mathbf{y}|=1} \mathbf{y} \cdot \nabla h(\mathbf{x} + \mathbf{y}tc) dS_y - \frac{1}{4\pi ct^2} \int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}') dV' + \frac{1}{4\pi t} \int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y}) dS_y \\ &= \frac{1}{4\pi ct^2} \int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}') dV' - \frac{1}{4\pi ct^2} \int_{V(\mathbf{x},t)} \nabla^2 h(\mathbf{x}') dV' + \frac{1}{4\pi t} \int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y}) dS_y \\ &= \frac{1}{4\pi t} \int_{S(\mathbf{x},t)} \nabla^2 h(\mathbf{y}) dS_y \\ &= \frac{c^2 t}{4\pi} \int_{|\mathbf{y}|=1} h(\mathbf{x} + \mathbf{y}tc) dS_y = c^2 \nabla^2 \phi. \end{aligned} \quad (86)$$

Thus we have shown that ϕ satisfies the wave equation.

Given these facts we may write down Kirchoff's solution to the initial value problem with initial data f, g :

$$\phi(\mathbf{x}, t) = \frac{1}{4\pi tc^2} \int_{S(\mathbf{x},t)} g(\mathbf{x}') dS' + \frac{\partial}{\partial t} \frac{1}{4\pi tc^2} \int_{S(\mathbf{x},t)} f(\mathbf{x}') dS'. \quad (87)$$

Although we have seen that the domain of dependence of a point in space at a future time is in fact a finite segment of the line in one dimension, the corresponding statement in 3D, that the domain of dependence is a finite region of 3-space, is false. The actual domain of dependence is the surface of a sphere of radius ct , centered at \mathbf{x} . This fact is known as *Huygen's principle*.

We note that the bursting balloon problem can be solved directly using the Kirchoff formula. A nice exercise is to compare this method with the 1D solution we gave above.

Moving sound sources give rise to different sound field depending upon whether or not the source is moving slower or faster than the speed of sound. In the latter case, a source moving to the left along the x -axis with a speed $U > c$ will produce sound waves having a conical envelope, see figure 2.

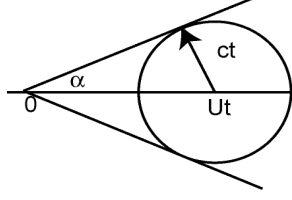


Figure 2 . Supersonic motion of a sound source.

Here $\sin \alpha = c/U = 1/M$ where $M = U/c$ is the *Mach number*. A moving of a slender body through a compressible fluid at supersonic speeds can be thought of as a sound source. The effect of the body is then confined to within the conical surface of figure 2. This surface is called the *Mach cone*.

5.4 Weakly nonlinear acoustics in 1D

We have seen that sound propagation in 1D involves the characteristics $x \pm ct = \text{constant}$, representing to directions of propagation. If a sound pulse traveling in one of these directions is followed, over time weak nonlinear effects can become important, and a nonlinear equation is needed to describe the compressive waves. In this section we shall derive the equation that replaces the simple linear wave equation $\phi_t \pm c\phi_x = 0$ associated with the two families of characteristics.

We shall suppose that the disturbance is moving to the right, i.e. is in linear theory a function of $x - ct$ alone. The characteristic coordinates

$$\xi = x - ct, \quad \eta = x + ct \quad (88)$$

can be used in place of x, t provided $c > 0$. Then

$$\frac{\partial}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}. \quad (89)$$

Thus with the linear theory our right-moving disturbance is annihilated by the operator

$$\frac{\partial}{\partial \eta} = \frac{1}{2} \frac{\partial}{c\partial t} + \frac{\partial}{\partial x}. \quad (90)$$

We shall be therefore looking a compressive wave which, owing to nonlinearity, has a nonzero but small variation with respect to η . The variation with respect to ξ will involve small effects, both from nonlinearity and from the viscous stresses.

If the variables are again $\rho_0 + \rho', p_0 + p'$, and u' , the exact conservation of mass equation is

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = -\frac{\partial(\rho' u')}{\partial x}. \quad (91)$$

To get the proper form of the momentum equation we expand the pressure as a function of ρ , assuming that we have a polytropic gas. With $p = h\rho^\gamma$ we have the Taylor series

$$p = p_0 + c^2 \rho' \frac{c^2(\gamma - 1)}{\rho_0} \frac{(\rho')^2}{2} + \dots \quad (92)$$

Here we have use $c^2 = \gamma k \rho^{\gamma-1}$. Thus the momentum equation takes the form, through terms quadratic in primed quantities,

$$\rho_0 \frac{\partial u'}{\partial t} + c^2 \frac{\partial \rho'}{\partial x} = -\rho' \frac{\partial u'}{\partial t} - \rho_0 u' \frac{\partial u'}{\partial x} - \frac{\gamma - 1}{\rho_0} c^2 \rho' \frac{\partial \rho'}{\partial x} + \frac{4\mu}{3} \frac{\partial^2 u'}{\partial x^2}. \quad (93)$$

Note that the viscous stress term comes from the difference $2\mu - \frac{2}{3}\mu$ in the coefficient of $\frac{\partial u'}{\partial x}$ in the 1D stress tensor. We assume here that μ is a constant.

To derive a nonlinear equation for the propagating disturbance we proceed in two steps. First, eliminate the ξ differentiations from the linear parts of the two equations. This will yield a equation with a first derivative term in η , along with the viscosity term and a collection of quadratic nonlinearities in u', ρ' . Then we use the approximate *linear* relation between u' and ρ' to eliminate ρ' in favor of u' in these terms. The result will be a nonlinear equation for u' in which all terms are small but comparable.

The linear relation used in the nonlinear terms comes from

$$\rho_0 \frac{\partial u'}{\partial t} = -c^2 \frac{\partial \rho'}{\partial x} \quad (94)$$

Since dependence upon η is weak, the last relation expressed in ξ, η variables becomes

$$-c\rho_0 \frac{\partial u'}{\partial \xi} = -c^2 \frac{\partial \rho'}{\partial \xi}, \quad (95)$$

so that

$$\rho' \approx \frac{\rho_0}{c} u' \quad (96)$$

in the nonlinear terms as well as in any derivative with respect to η .

Now in characteristic coordinates the linear parts of the equations take the forms

$$\frac{\partial}{\partial \xi}(-c\rho' + \rho_0 u') + \frac{\partial}{\partial \eta}(c\rho' + \rho_0 u') = -\frac{\partial(\rho' u')}{\partial x}. \quad (97)$$

$$\frac{\partial}{\partial \xi}(-c\rho_0 u' + c^2 \rho') + \frac{\partial}{\partial \eta}(c\rho_0 u' + c^2 \rho') = \dots, \quad (98)$$

where the RHS consists of nonlinear and viscous terms. Dividing the second of these by c and adding the two equations we get

$$2 \frac{\partial}{\partial \eta}(c\rho' + \rho_0 u') = -\frac{\partial(\rho' u')}{\partial x} - \frac{1}{c} \rho' \frac{\partial u'}{\partial t} - \frac{\rho_0}{c} u' \frac{\partial u'}{\partial x} - \frac{\gamma - 1}{\rho_0} c \rho' \frac{\partial \rho'}{\partial x} + \frac{4\mu}{3c} \frac{\partial^2 u'}{\partial x^2}. \quad (99)$$

Since the LHS here involves now only the η derivative, we may use (96) to eliminate ρ' , and similarly with all terms on the RHS. Also we express x and t derivatives on the RHS in terms of ξ . Thus we have

$$4\rho_0 \frac{\partial u'}{\partial \eta} = -2 \frac{\rho_0}{c} u' \frac{\partial u'}{\partial \xi} + \frac{\rho_0}{c} u' \frac{\partial u'}{\partial \xi} - + \frac{\rho_0}{c} u' \frac{\partial u'}{\partial \xi} - \frac{(\gamma - 1)\rho_0}{c} u' \frac{\partial u'}{\partial \xi} + \frac{4\mu}{3c} \frac{\partial^2 u'}{\partial \xi^2}. \quad (100)$$

Thus

$$2c \frac{\partial u'}{\partial \eta} + \frac{\gamma + 1}{2} u' \frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0} \frac{\partial^2 u'}{\partial \xi^2} = 0. \quad (101)$$

Now

$$\frac{\partial}{\partial \eta} = \frac{1}{2c} \left(\frac{\partial}{\partial t} + c \frac{\partial}{\partial x} \right), \quad (102)$$

so

$$\frac{\partial u'}{\partial t}|_x + c \frac{\partial u'}{\partial x}|_t + \frac{\gamma + 1}{2} u' \frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0} \frac{\partial^2 u'}{\partial \xi^2} = 0. \quad (103)$$

Now this involves a linear operator describing the time derivative relative to an observer moving with the speed c . The linear operator is now the time derivative holding ξ fixed. Thus

$$\frac{\partial u'}{\partial t}|_\xi + \frac{\gamma + 1}{2} u' \frac{\partial u'}{\partial \xi} - \frac{2\mu}{3\rho_0} \frac{\partial^2 u'}{\partial \xi^2} = 0. \quad (104)$$

The velocity perturbation u' , we emphasize, is that relative to the fluid at rest at infinity. Moving with the wave the gas is seen to move with velocity $u = u' - c$ or u' really denotes $u + c$ where u is the velocity seen by the moving observer.

What we have in (104) is the viscous form of *Burgers' equation*. It is a nonlinear wave equation incorporating viscous dissipation but not dispersion. By suitable scaling it may be brought into the form

$$u_t + uu_x - \nu u_{xx} = 0. \quad (105)$$

If the viscous term is dropped we have the inviscid Burgers wave equation,

$$u_t + uu_x = 0. \quad (106)$$

This equation is much studied as a prototypical nonlinear wave equation. We review the method of characteristics for such equations in the next section.

6 Nonlinear waves in one dimension

The simplest scalar wave equation can be written in the conservation form

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} F(u) = 0, \quad (107)$$

equivalent to

$$\frac{\partial u}{\partial t} + v(u) \frac{\partial u}{\partial x} = 0, \quad v(u) = F'(u). \quad (108)$$

The last equation can be regarded as stating that an observer moving with the velocity $v(u)$ observes that u does not change. The particle path of the observer is called a *characteristic curve*. Since u is constant on the characteristic and the velocity v is a function of u alone we see that the characteristic is a straight line in the x, t - plane. If $u(x, 0) = u_0(x)$, the characteristics are given by the family

$$x = v(u_0(x_0))t + x_0. \quad (109)$$

Here x_0 acts like a Lagrangian coordinate, marking the intersection of the characteristic with the initial line $t = 0$.

As an example of the solution of the initial-value problem using characteristics, consider the equation

$$\frac{\partial u}{\partial t} + u^2 \frac{\partial u}{\partial x} = 0, \quad u_0(x) = \begin{cases} 0, & \text{if } x < 0, \\ x, & \text{if } 0 < x < 1, \\ 1, & \text{if } x > 1 \end{cases} \quad (110)$$

First observe that the characteristics are vertical line in $x < 0$, so that $u = 0$ in $x < 0, t > 0$. Similarly the characteristics are the line $x = t + x_0$ when $x_0 > 1$, so that $u = 1$ when $x > 1 + t$. Solving $x = x_0^2 t + x_0$ for $x_0(x, t)$, we arrive at the following solution in the middle region $0 < x < 1 + t$:

$$u(x, t) = \frac{-1 + \sqrt{1 + 4xt}}{2t}. \quad (111)$$

Now we modify the initial condition to

$$u_0(x) = \begin{cases} 0, & \text{if } x < 0, \\ x/\epsilon, & \text{if } 0 < x < \epsilon, \\ 1, & \text{if } x > \epsilon \end{cases} \quad (112)$$

The solution is then, since the characteristics in the middle region are $x = (x_0/\epsilon)^2 t + x_0$

$$u(x, t) = \frac{\epsilon}{2t} \left[-1 + \sqrt{1 + 4xt/\epsilon^2} \right]. \quad (113)$$

letting $\epsilon \rightarrow 0$ in (113) we obtain

$$u \rightarrow \sqrt{\frac{x}{t}}. \quad (114)$$

This solution, existing in the wedge $0 < x/t < 1$ of the x, t -plane, is called an *expansion fan*. Given the discontinuous initial condition

$$u = \begin{cases} 0, & \text{if } x < 0, \\ 1, & \text{if } x > 0, \end{cases} \quad (115)$$

we can solve for the expansion fan directly by noting that u must be a function of $\eta = x/t$. Substituting $u = f(\eta)$ in our equation, we obtain

$$-\eta f' + f^2 f' = 0, \quad (116)$$

implying $f = \pm\sqrt{\eta}$. The positive sign is needed to make the solution continuous at the edges of the fan.

6.1 Dynamics of a polytropic gas

We have the following equation for a polytropic gas in one dimension, in the absence of dissipative processes and assuming constant entropy:

$$u_t + uu_x + \frac{c^2}{\rho} \rho_x, \quad \rho_t + u\rho_x + \rho u_x = 0. \quad (117)$$

Here $c^2 = k\gamma\rho^{\gamma-1}$. If we define the column vector $[u \ \rho]^T = w$, the system may be written $w_t + A \cdot w_x$ where

$$A = \begin{pmatrix} u & c^2/\rho \\ \rho & u \end{pmatrix}. \quad (118)$$

We now try to find analogs of the characteristic lines $x \pm ct = \text{constant}$ which arose in acoustics in one space dimension. We want to find curves on which some physical quantity is invariant. Suppose that v is a right eigenvector of A^T (transpose of A), $A^T \cdot v = \lambda v$. We want to show that the eigenvalue λ plays a role analogous to the acoustic sound velocity.

Indeed, we see that

$$v^T \cdot [w_t + A \cdot w_x] = v^T \cdot w_t + A^T \cdot v \cdot w_x = v^T \cdot w_t + \lambda v^T \cdot w_x = 0. \quad (119)$$

Now suppose that we can find an integrating factor ϕ such that $\phi v^T \cdot dw = dF$. Then we would have

$$F_t + \lambda F_x = 0. \quad (120)$$

Thus $dx/dt = \lambda$ would define a characteristic curve in the x, t - plane on which $F = \text{constant}$. The quantity F is called a *Riemann invariant*.

Thus we solve the eigenvalue equation

$$\det(A^T - \lambda I) = \left| \begin{pmatrix} u - \lambda & \rho \\ c^2/\rho & u - \lambda \end{pmatrix} \right| = 0. \quad (121)$$

Then $(u - \lambda)^2 = c^2$, or

$$\lambda = u \pm c \equiv \lambda_{\pm}. \quad (122)$$

We see that the characteristic speeds are indeed related to sound velocity, but now altered by the doppler shift introduced by the fluid velocity. (Unlike light through space, the speed of sound does depend upon the motion of the observer. Sound moves relative to the compressible fluid in which it exists.)

Thus the following eigenvectors are obtained:

$$\lambda_+ : \begin{pmatrix} -c & \rho \\ c^2/\rho & -c \end{pmatrix} v_+ = 0, \quad v_+^T = [\rho \ c], \quad (123)$$

$$\lambda_- : \begin{pmatrix} c & \rho \\ c^2/\rho & c \end{pmatrix} v_- = 0, \quad v_-^T = [\rho \ -c], \quad (124)$$

We now choose ϕ :

$$\phi[\rho \ \pm c] \begin{bmatrix} du \\ d\rho \end{bmatrix} = dF_{\pm}. \quad (125)$$

Since c is a function of ρ we see that we may take $\phi = 1/\rho$, to obtain

$$F_{\pm} = u \pm \int \frac{c}{\rho} d\rho, \quad (126)$$

which may be brought into the form

$$F_{\pm} = u \pm \frac{2}{\gamma - 1}c. \quad (127)$$

Thus we find that the Riemann invariants $u \pm \frac{2}{\gamma - 1}c$ are constant on the curves $\frac{dx}{dt} = u \pm c$:

$$\left[\frac{\partial}{\partial t} + (u \pm c) \frac{\partial}{\partial x} \right] \left[u \pm \frac{2}{\gamma - 1}c \right] = 0. \quad (128)$$

6.2 Simple waves

Any region of the x, t -plane which is adjacent to a region where fluid variables are constant (i.e. a region at a constant state), but which is not itself a region of constant state, will be called a *simple wave* region, or SWR. The characteristic families of curves associated with λ_{\pm} will be denoted by C_{\pm} . Curves of both families will generally propagate through a region. In a simple wave region one family of characteristics penetrates into the region of constant state, so that one of the two invariants F_{\pm} will be known to be constant over a SWR. Suppose that F_{-} is constant over the SWR. Then any C_{+} characteristic in the SWR not only carries a constant value of F_{+} but also a constant value of F_{-} , and this implies a constant value of $u + c$ (see the definitions (127) of F_{\pm}). Thus *in a SWR where F_{-} is constant the C_{+} characteristics are straight lines*, and similarly for the C_{-} characteristics over a SWR where F_{+} is constant.

Let us suppose that a simple wave region involving constant F_{-} involves $u > 0$, so fluid particles move upward in the x, t -plane. All of the C_{+} characteristics have positive slope. They may either converge or diverge. In the latter case we have the situation shown in figure 3(a). Since $u + c > u$, fluid particles must cross the C_{+} characteristics from right to left. Moving along this path, a fluid particle experiences steadily decreasing values of $u + c$. We assume now that $\gamma > 1$. Since $u = \frac{2c}{\gamma - 1} + \text{constant}$ by the constancy of F_{-} , we see that in this motion of the fluid particle c , and hence ρ , is decreasing. Thus the fluid is becoming less dense, or expanding. We have in figure 3(a) what we shall call a *forward-facing expansion wave*. Similarly in figure 3(b) F_{+} is constant in the SWR, and $u - c$ is constant on each C_{-} characteristic. These are again an expansion waves, and we term them backward-facing. Forward and backward-facing compression waves are similarly obtained when C_{+} characteristics converge and C_{-} characteristics diverge.

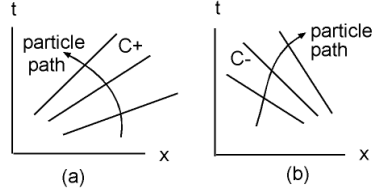


Figure 3. Simple expansion waves. (a) Forward-facing (b) Backward-facing.

6.3 Example of a SWR: pull-back of a piston

We consider the movement of a piston in a tube with gas to the right, see figure 4. The motion of the piston is described by $x = X(t)$, the movement being to the left, $dX/dt < 0$. If we take $X(t) = -at^2/2$, then $u = -at$ on the piston. We assume that initially $u = 0, \rho = \rho_0$ in the tube.

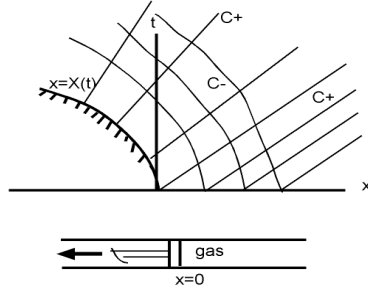


Figure 4. Pull-back of a piston, illustrating a simple wave region.

On the C_- characteristics, we have $u - \frac{2c}{\gamma-1} = F_- = \frac{-2c_0}{\gamma-1}$, or

$$c = c_0 + \frac{\gamma-1}{2}u. \quad (129)$$

Also on C_+ characteristics we have $u + \frac{2c}{\gamma-1}$ constant. By this fact and (129) we have $2u + \frac{2c_0}{\gamma-1}$ is constant on the C_+ characteristics. But since $u = u_p = -at$ at the piston surface, this determines the constant value of u . Let the C_+ characteristic in question intersect the piston path at $t = t_0$. Then the equation of this characteristic is

$$\frac{dx}{dt} = u + c = \frac{\gamma+1}{2}u + c_0 = -\frac{\gamma+1}{2}at_0 + c_0. \quad (130)$$

Thus

$$x = -a\frac{\gamma+1}{2}t_0t + \frac{a\gamma}{2}t_0^2. \quad (131)$$

If we solve the last equation for $t_0(X, T)$ we obtain

$$t_0 = \frac{1}{a\gamma} \left[\frac{at(\gamma+1)}{2} + \sqrt{\left(\frac{at(\gamma+1)}{2}\right)^2 + 2xa\gamma} \right]. \quad (132)$$

Then $u = -at_0(x, t)$ in the simple wave region, c being given by (129).

Note that according to (129), $c = 0$ when $t = t^*$, where $at^* = \frac{2}{\gamma-1}c_0$. This piston speed is the limiting speed the gas can obtain. For $t > t^*$ the piston pulls away from a vacuum region bounded by an interface moving with speed $-at^*$.

If we consider the case of instantaneous motion of the velocity with speed u_p , the C_+ characteristics emerge from the origin as an expansion fan. Their equation is

$$\frac{x}{t} = u + c = c_0 + \frac{\gamma+1}{2}u, \quad (133)$$

so that

$$u = \frac{2}{\gamma+1} \left[\frac{x}{t} - 2c_0 \right]. \quad (134)$$

To compute the paths $\xi(t)$ of fluid particles in this example, we must solve

$$\frac{d\xi}{dt} = \frac{2}{\gamma+1} \left[\frac{\xi}{t} - 2c_0 \right]. \quad (135)$$

A particle begins to move with the rightmost wave of the expansion fan, namely the line $x = c_0t$, meets the initial particle position. Thus (135) must be solved with the initial condition $\xi(t_0) = c_0t_0$. The solution is

$$\xi(t) = \frac{-2c_0}{\gamma-1}t + c_0t_0 \frac{\gamma+1}{\gamma-1} (t/t_0)^{\frac{2}{\gamma+1}}. \quad (136)$$

For the location of the C_- characteristics we must solve

$$\frac{dx}{dt} = u - c = \frac{3-\gamma}{2}u - c_0 = \frac{3-\gamma}{\gamma+1}(x/t - c_0) - c_0, \quad (137)$$

with the initial condition $x(t_0) = c_0t_0$. There results

$$x(t) = \frac{-2c_0}{\gamma-1}t + c_0t_0 \frac{\gamma+1}{\gamma-1} (t/t_0)^{\frac{3-\gamma}{1-\gamma}}. \quad (138)$$

7 Linearized supersonic flow

We have seen above that 2D irrotational inviscid homentropic flow of a polytropic gas satisfies the system

$$(c^2 - \phi_x^2)\phi_{xx} + (c^2 - \phi_y^2)\phi_{yy} - 2\phi_x\phi_y\nabla^2\phi = 0, \quad (139)$$

$$\phi_x^2 + \phi_y^2 + \frac{2}{\gamma-1}c^2 = \text{constant}. \quad (140)$$

We are interested in the motion of thin bodies which do not disturb the ambient fluid very much, The assumption of small perturbations, and the corresponding linearized theory of compressible flow, allows us to consider some steady flow problems of practical interest which are analogs of sound propagation problems.

We assume that the air moves with a speed U past the body, from left to right in the direction of the x-axis. Then the potential is taken to have the form

$$\phi = U_0x + \phi', \quad (141)$$

where $|\phi'_x| \ll U_0$. It is easy to derive the linearized form of (139), since the second-derivative terms must be primed quantities. The other factors are then evaluated at the ambient conditions, $c \approx c_0$, $\phi_x \approx U$. Thus we obtain

$$(M^2 - 1)\phi'_{xx} - \phi'_{yy}. \quad (142)$$

Here

$$M = U_0/c_0 \quad (143)$$

is the *Mach number* of the ambient flow. We note that in the linear theory the pressure is obtained from

$$U_0 \frac{\partial u'}{\partial x} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x}, \quad (144)$$

or

$$p' \approx -U_0 \rho_0 \phi'_x. \quad (145)$$

We now drop the prime from ϕ' . The density perturbation is then $\rho' = -c_0^{-2} U_0 \rho \phi_x$.

7.1 Thin airfoil theory

We consider first the 2D supersonic flow over a thin airfoil.

Linearized supersonic flow results when $M > 1$, linearized subsonic flow when $M < 1$. The *transonic regime* $M \approx 1$ is special and needs to be examined as a special case.

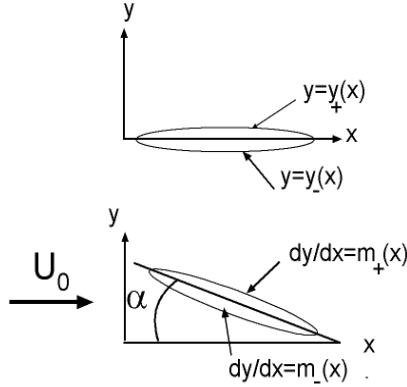


Figure 5. Thin airfoil geometry.

We show the geometry of a thin airfoil in figure 5. We assume that the slopes dy_{\pm}/dx and the angle of attack α are small. In this case

$$m_{\pm}(x) \approx -\alpha + dy_{\pm}/dx. \quad (146)$$

let the chord of the airfoil be l , so we consider $0 < x < l$. We note that

$$\frac{1}{l} \int_0^l (m_+ + m_-) dx = -2\alpha, \quad \frac{1}{l} \int_0^l (m_+^2 + m_-^2) dx = -2\alpha^2 + \frac{1}{l} \int_0^l (y_+^2 + y_-^2) dx. \quad (147)$$

The analysis now makes use the following fact analogous to the linear wave equation in 1D: the linear operator factors as

$$[\sqrt{M^2 - 1} \frac{\partial}{\partial x} - \frac{\partial}{\partial y}] [\sqrt{M^2 - 1} \frac{\partial}{\partial x} + \frac{\partial}{\partial y}]. \quad (148)$$

Thus $\phi = f(x - y\sqrt{M^2 - 1}) + g(x + y\sqrt{M^2 - 1})$, where f, g are arbitrary functions. We now need to use physical reasoning to choose the right form of solution. In linearized supersonic flow past an airfoil the disturbances made by the foil propagate out relative to the fluid at the speed of sound, but are simultaneously carried downstream with speed U_0 . In supersonic flow the foil cannot

therefore cause disturbances of the fluid upstream of the body. Consequently the characteristic lines $x \pm y\sqrt{M^2 - 1} = \text{constant}$, which carry disturbances away from the foil, must always point downstream. Thus in the half space above the foil the correct choice is $\phi = f(x - y\sqrt{M^2 - 1})$, while in the space below it the correct choice is $\phi = g(x + y\sqrt{M^2 - 1})$. To determine these functions, we must make the flow tangent to the foil surface. Since we are dealing with thin airfoils and small angles, the condition of tangency can be applied, approximately, at $y = 0$. Thus we have the tangency conditions

$$\frac{\phi_y}{U_0}|_{y=0+} = m_+(x) = -\sqrt{M^2 - 1}U_0^{-1}f'(x), \quad (149)$$

$$\frac{\phi_y}{U_0}|_{y=0-} = m_-(x) = \sqrt{M^2 - 1}U_0^{-1}g'(x), \quad (150)$$

Of interest to engineers is the lift and drag of a foil. To compute these we first need the pressures

$$p'_+(x) = -U_0\rho_0u'(x, 0+) = -U_0\rho_0f'(x), \quad p'_-(x) = -U_0\rho_0g'(x). \quad (151)$$

This yields

$$p'_\pm = \pm \frac{U_0^2\rho_0}{\sqrt{M^2 - 1}}(-\alpha + dy_\pm/dx). \quad (152)$$

Then

$$\text{Lift} = \int_0^l (p'_- - p'_+)dx = \frac{2\alpha\rho_0U_0^2l}{\sqrt{M^2 - 1}}, \quad (153)$$

$$\text{Drag} = \int_0^l (p'_+m_+ - p'_-m_-)dx = \frac{\rho_0U_0^2l}{\sqrt{M^2 - 1}}[2\alpha^2 + \frac{1}{l} \int_0^l [(dy_+/dx)^2 + (dy_-/dx)^2]dx. \quad (154)$$

Note that now inviscid theory gives a positive drag. We recall that for incompressible potential flow we obtained zero drag (D'Alembert's paradox). In supersonic flow, the characteristics carry finite signals to infinity. In fact the disturbances are being created so that the rate of increase of kinetic energy per unit time is just equal to the drag times U_0 . This drag is often called *wave drag* because it is associated with characteristics, usually called in this context *Mach waves*, which propagate to infinity.

What happens if we solve for compressible flow past a body in the subsonic case $M < 1$? In the case of thin airfoil theory, it is easy to see that we must get zero drag. The reason is that the equation we are now solving may be written $\phi_{xx} + \phi_{\bar{y}\bar{y}} = 0$ where $\bar{y} = \sqrt{1 - M^2}y$. The boundary conditions are at $y = \bar{y} = 0$ so in the new variables we have a problem equivalent to that of an incompressible potential flow.

In fact compressible potential flow past any finite body will give zero drag so long as the flow field velocity never exceeds the local speed of sound, i.e. the fluid stays locally subsonic everywhere. In that case no shock waves can form, there is no dissipation, and D'Alembert's paradox remains.

7.2 Slender body theory

Another case of interest is the steady supersonic flow past a slender body of revolution. If the ambient flow is along the z -axis in cylindrical polar coordinates x, r , the body we consider is a slender body of revolution about the z -axis. It is easy to show that the appropriate wave equation (coming from the linearized equations $\rho_0 U_0 \frac{\partial u'}{\partial z} + c_0^2 \frac{\partial \rho'}{\partial z} = 0$, $U_0 \frac{\partial \rho'}{\partial z} + \rho_0 \nabla \cdot \mathbf{u}' = 0$), is

$$\beta^2 \phi_{zz} - \phi_{rr} - \frac{1}{r} \phi_r = 0, \quad \beta = \sqrt{M^2 - 1}. \quad (155)$$

To find a fundamental solution of this equation, note that

$$a^2 \phi_{xx} + b^2 \phi_{yy} + c^2 \phi_{zz} = 0 \quad (156)$$

clearly has a “sink-like” solution $[(x/a)^2 + (y/b)^2 + (z/c)^2]^{-1}$, equivalent to the simple sink solution (-4π time the fundamental solution) $1/\sqrt{(x^2 + y^2 + z^2)}$ of Laplace’s equation in 3D. This holds for arbitrary complex numbers a, b, c . It follows that a solution of (155) is given by

$$S(z, r) = \frac{1}{\sqrt{z^2 - \beta^2 r^2}}. \quad (157)$$

Note that this is a real quantity only if $\beta r < z$, where S is singular. We therefore want to complete the definition of S by setting

$$S(z, r) = 0, \quad \beta r > z. \quad (158)$$

Suppose now that we superimpose these solutions by distributing them on the interval $(0, l)$ of the z -axis,

$$\phi = \int_0^l \frac{f(\zeta)}{\sqrt{(z - \zeta)^2 - \beta^2 r^2}} d\zeta. \quad (159)$$

However notice that if we are interested in the solution on the surface $z - \beta r = C$, then there can be no contributions from values of ζ exceeding C . We therefore propose a potential

$$\phi(z, r) = \begin{cases} \int_0^{z - \beta r} \frac{f(\zeta)}{\sqrt{(z - \zeta)^2 - \beta^2 r^2}} d\zeta, & \text{if } 0 < z - \beta r < l, \\ \int_0^l \frac{f(\zeta)}{\sqrt{(z - \zeta)^2 - \beta^2 r^2}} d\zeta, & \text{if } z - \beta r > l. \end{cases} \quad (160)$$

where we now require $f(0) = 0$.

We can in fact verify that (160) gives us a solution of (155) for any admissible f , but will leave this verification as a problem.

Consider now the behavior of ϕ near the body. When r is small the main contribution comes from the vicinity of $\zeta = z$, so we may extract $f(z)$ and use the change of variables $\zeta = z - \beta r \cosh \lambda$ to obtain

$$\phi \approx f(z) \int_0^{\cosh^{-1}(z/\beta r)} d\lambda = f(z) \cosh^{-1}(z/\beta r) \sim f(z) \log(2z/\beta r). \quad (161)$$

Now let the body be described by $r = R(z), 0 < z < l$. The tangency condition is then

$$r \frac{\phi_r}{U_0} \Big|_{r=R} = R \frac{dR}{dz} \approx -f(z)/U_0. \quad (162)$$

If $A(a)$ denotes cross-sectional area, then we have

$$f(z) = -\frac{1}{2\pi} U_0 \frac{dA}{dz}. \quad (163)$$

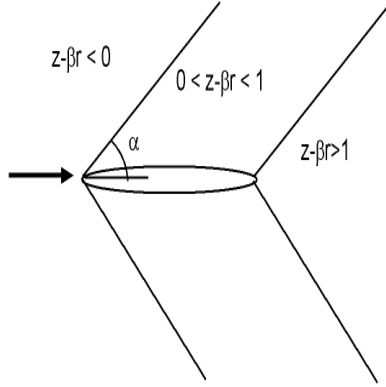


Figure 6. Steady supersonic flow past a slender body of revolution. Here $\tan \alpha = 1/\sqrt{M^2 - 1}$.

The calculation of drag is a bit more complicated here, and we give the result for the case where $R(0) = R(l) = 0$. Then

$$\text{Drag} = \frac{\rho_0 U_0^2}{4\pi} \int_0^l \int_0^l A''(z) A''(\zeta) \frac{1}{\log |z - \zeta|} dz d\zeta. \quad (164)$$

The form of this emphasizes the importance of having a smooth distribution of area in order to minimize drag.

7.2.1 An alternative formulation and proof of the drag formula

To prove (164) it is convenient to reformulate the problem in terms of a streamfunction. We go back to the basic equations for steady homentropic potential flow in cylindrical polar coordinates.

$$\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r} = 0, \quad \frac{\partial r \rho u_z}{\partial z} + \frac{\partial r \rho u_r}{\partial r} = 0, \quad (165)$$

$$u_z^2 + u_r^2 + \frac{2}{\gamma - 1} c^2 = \text{constant}. \quad (166)$$

From the second of (165) we introduce the streamfunction ψ ,

$$r\rho u_z = \frac{\partial\psi}{\partial r}, r\rho u_r = -\frac{\partial\psi}{\partial z}. \quad (167)$$

We then expand the equations as follows:

$$\psi = \frac{U_0}{2}r^2 + \psi', \rho = \rho_0 + \rho', \quad (168)$$

and linearize. The result is the equation for ψ' ,

$$r \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial\psi'}{\partial r} - \beta^2 \frac{\partial^2\psi'}{\partial z^2} = 0. \quad (169)$$

Now the boundary condition in terms of the stream function is that ψ equal zero on the slender body $r = R(z)$. Approximately, this gives

$$\rho_0 \frac{U_0}{2} r^2 + \psi'(z, 0) \approx 0. \quad (170)$$

We also want no disturbance upstream, so ψ' and ψ'_z should vanish on $z = 0, r > 0$. A solution of this problem is given by

$$\psi' = \begin{cases} -\frac{\rho_0 U_0}{2\pi} \int_0^{z-\beta r} \sqrt{(z-\zeta)^2 - \beta^2 r^2} A'' d\zeta, & \text{if } z \geq \beta r, \\ 0, & \text{if } z < \beta r. \end{cases} \quad (171)$$

It is easy to see that the equation and upstream conditions are satisfied under the conditions that $R(0) = 0$. For the boundary condition we have

$$\psi'(z, 0) = -\frac{\rho_0 U_0}{2\pi} \int_0^z (z-\zeta) A'' d\zeta = -\frac{\rho_0 U_0}{2\pi} \int_0^z A' d\zeta = -\rho_0 \frac{U_0}{2} R^2. \quad (172)$$

Now the drag is given by

$$D = \int_0^l p' A'(\zeta) d\zeta, \quad (173)$$

where the linear theory gives $p' \approx -\rho_0 U_0 u'_z$. However it turns out that the u_r velocity components become sufficiently large near the body to make a leading order contribution. Thus we have

$$p' \approx -\rho_0 U_0 u'_z - \frac{\rho_0}{2} (u'_r)^2 + \dots \quad (174)$$

We note that

$$u'_z = \left(\frac{1}{\rho r} \frac{\partial\psi}{\partial r} \right)' \approx \frac{1}{\rho_0 r} \frac{\partial\psi'}{\partial r} - \frac{U_0}{\rho_0 c_0^2} p', \quad (175)$$

from which we have

$$-\beta^2 u'_z = \frac{1}{\rho_0 r} \frac{\partial\psi'}{\partial r}. \quad (176)$$

Thus

$$u'_z = -\frac{U_0}{2\pi} \int_0^{z-\beta r} \frac{A''(\zeta)}{\sqrt{(z-\zeta)^2 - \beta^2 r^2}} d\zeta. \quad (177)$$

Similarly

$$u'_r = \frac{U_0}{2\pi r} \int_0^{z-\beta r} (z-\zeta) \frac{A''(\zeta)}{\sqrt{(z-\zeta)^2 - \beta^2 r^2}} d\zeta. \quad (178)$$

We see by letting $r \rightarrow R \approx 0$ in (178) that

$$u'_r \approx \frac{U_0}{2\pi} A'(z)/R(z). \quad (179)$$

Also

$$u'_z \approx -\frac{U_0}{2\pi} \left[A''(z-\beta r) \cosh^{-1} \frac{z}{\beta r} + \int_0^{z-\beta r} \frac{A''(\zeta) - A''(z-\beta r)}{\sqrt{(z-\zeta)^2 - \beta^2 r^2}} d\zeta \right] \quad (180)$$

$$\approx -\frac{U_0}{2\pi} \left[A'' \log \frac{2z}{\beta R} - \int_0^z \frac{A''(z) - A''(\zeta)}{z-\zeta} d\zeta \right]. \quad (181)$$

We are now in a position to compute D :

$$D = \frac{\rho_0 U_0^2}{2\pi} \int_0^l A'(z) \int_0^{z-\beta R} \left[A''(z) \log \frac{2z}{\beta r} - \int_0^z \frac{A''(z) - A''(\zeta)}{z-\zeta} d\zeta - \frac{1}{4} A^{-1}(z) (A')^2(z) \right] dz. \quad (182)$$

After an integration by parts and a cancelation we have

$$D = \frac{\rho_0 U_0^2}{2\pi} \int_0^l A'(z) \left[A''(z) \log z - \int_0^z \frac{A''(z) - A''(\zeta)}{z-\zeta} d\zeta \right] dz. \quad (183)$$

Our last step is to show that (183) agrees with (164). Now if $B(z) = A'(z)$,

$$\begin{aligned} \int_0^l \int_0^l B'(z) B'(\zeta) \log |z-\zeta| d\zeta dz &= 2 \int_0^l B'(z) \int_0^z B'(\zeta) \log |z-\zeta| d\zeta dz \\ &= 2 \int_0^l B'(z) B(z) \log z dz + 2 \int_0^l B'(z) \int_0^z \frac{B(z) - B(\zeta)}{z-\zeta} d\zeta dz \\ &= -2 \int_0^l B'(z) B(z) \log z dz - 2 \int_0^l B(z) \int_0^z \frac{B'(z) - B'(\zeta)}{z-\zeta} d\zeta dz, \end{aligned} \quad (184)$$

which proves the agreement of the two expressions. Here we have used

$$\begin{aligned} \frac{d}{dz} \int_0^z \frac{B(z) - B(\zeta)}{z-\zeta} d\zeta &= \frac{B(z) - B(\zeta)}{z-\zeta} \Big|_{\zeta} = z + \int_0^z \frac{(z-\zeta)B'(z) - B(z) + B(\zeta)}{(z-\zeta)^2} d\zeta \\ &= B'(z) + \left[\frac{(z-\zeta)b'(z) - B(z) + B(\zeta)}{(z-\zeta)^2} \right]_0^z + \int_0^z \frac{B'(z) - B'(\zeta)}{z-\zeta} d\zeta \\ &= \int_0^z \frac{B'(z) - B'(\zeta)}{z-\zeta} d\zeta + B(z)/z. \end{aligned} \quad (185)$$

8 Shock waves

8.1 Scalar case

We have seen that the equation $u_t + uu_x = 0$ with a initial condition $u(x, 0) = 1 - x$ on the segment $0 < x < 1$ produces a family of characteristics

$$x = (1 - x_0)t + x_0. \quad (186)$$

This family of lines intersects at $(x, t) = (1, 1)$. If the initial condition is extended as

$$u(x, 0) = \begin{cases} 1, & \text{if } x < 0, \\ 0, & \text{if } x > 1, \end{cases} \quad (187)$$

we see that at $t = 1$ a discontinuity develops in u as a function of x . We thus need to study how such discontinuities propagate for later times as *shock waves*. We study first the general scalar wave equation in conservation form, $u_t + (F(u))_x = 0$. This equation is assumed to come from a conservation law of the form

$$\frac{d}{dt} \int_a^b u dx = F(u(a, t)) - F(u(b, t)). \quad (188)$$

Suppose now that in fact there is a discontinuity present at position $\xi(t) \in (a, b)$. Then we study the conservation law by breaking up the interval so that differentiation under the integral sign is permitted:

$$\frac{d}{dt} \left[\int_a^{\xi} u dx + \int_{\xi}^b u dx \right] = F(u(a, t)) - F(u(b, t)). \quad (189)$$

Now differentiating under the integral and using the wave equation to eliminate the time derivatives of u we obtain

$$\frac{d\xi}{dt} [u(\xi+, t) - u(\xi-, t)] = F(u(\xi+, t)) - F(u(\xi-, t)). \quad (190)$$

Thus we have an expression for the propagation velocity of the shock wave:

$$\frac{d\xi}{dt} = \frac{[F]_{x=\xi}}{[u]_{x=\xi}}, \quad (191)$$

where here $[\cdot]$ means “jump in”. The direction you take the jump is immaterial provided that you do the same in numerator and denominator.

Example: Let $u_t + uu_x = 0$,

$$u(x, 0) = \begin{cases} 0, & \text{if } x < -1, \\ 1 + x, & \text{if } -1 < x < 0, \\ 1 - 2x, & \text{if } 0 < x < 1/2. \end{cases} \quad (192)$$

The characteristic family associated with the interval $-1 < x < 0$ is $x = (1 + x_0)t + x_0$, while that of $0 < x < 1/2$ is $x = (1 - 2x_0)t + x_0$. The shock first

occurs at $(x, t) = (1/2, 1/2)$. To the right of the shock $u = 0$, while to the left the former family gives

$$u = 1 + x_0(x, t) = 1 + \frac{x - t}{1 + t} = \frac{1 + x}{1 + t}. \quad (193)$$

Then

$$\frac{d\xi}{dt} = \frac{1}{2}[u(\xi-, t) + u(\xi+, t)] = \frac{1}{2} \frac{\xi + 1}{1 + t}, \quad \xi(1/2) = 1/2. \quad (194)$$

Thus

$$\xi(t) = \sqrt{3/2} \sqrt{1 + t} - 1. \quad (195)$$

We show the x, t -diagram for this in figure 7.

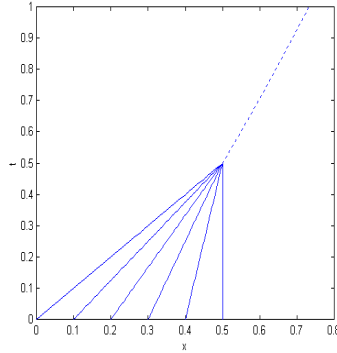


Figure 7. Example of shock formation and propagation.

8.2 A cautionary note

One peculiarity of shock propagation theory is that it is strongly tied to the physics of the problem. Suppose that $u \geq 0$ solves $u_t + uu_x = 0$. Then it will also solve $v_t + [G(v)]_x = 0$ where $G = \frac{2}{3}v^{3/2}$ and $v = u^2$. In the former case the shock wave propagation speed is $\frac{1}{2}(u_+ + u_-)$, while in the latter it is $\frac{2}{3}(u_+^2 + u_+u_- + u_-^2)/(u_+ + u_-)$, which is different. What's going on??

The point is that $u_t + uu_x = 0$ is based fundamentally on a conservation law involving $F(u) = u^2/2$. In actual physical problems the conservation laws will be known and have to be respected. Another way to say this is that equivalent partial differential equations can arise from different conservation laws. It is the conservation law that determines the relevant shock velocity however.

We illustrate this with a simple example from the continuum theory of traffic flow. Consider a single-lane highway with $n(x, t)$ cars per mile as the traffic density. The cars are assumed to move at a speed determined by the local density, equal to $u = U(1 - n/n_0)$ where U is the maximum velocity and n_0 is the density of full packing and zero speed. The flux of cars is the $F(n) = nu = Un(1 - n/n_0)$, and the corresponding conservation of car number yields

the PDE $n_t + [F(n)]_x = 0$. This is equivalent to $v_t + [G(v)]_x = 0$ where $v = n^2$ and $G = U[v - \frac{2}{3n_0}v^{3/2}]$. However the conservation law associated with v, G makes no physical sense. We know how the speed of the cars depends upon n , and conservation of number (if indeed that is what happens) dictates the former conservation law. Note that if the square of density was somehow what was important in the conservation of mass, we would end up with a conservation of mass equation $\frac{\partial \rho^2}{\partial t} + \nabla \cdot (\rho^2 \mathbf{u}) = 0$.

8.3 The stationary normal shock wave in gas dynamics

We now want to consider a stationary planar shock in gas dynamics, without viscosity or heat conduction. We assume that constant conditions prevail on either side of the shock denoted by the subscripts 1, 2, see figure 8.

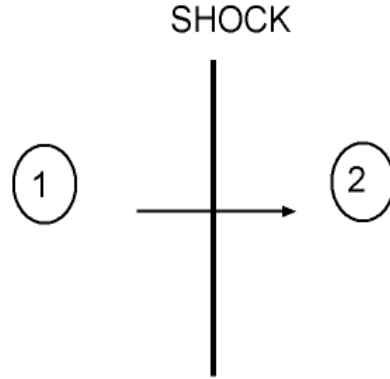


Figure 8. The stationary normal shock wave.

We have the following conservation laws:

$$\text{Mass : } \rho_1 u_1 = \rho_2 u_2. \quad (196)$$

$$\text{Momentum : } p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2. \quad (197)$$

Recall the following form of the energy equation:

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) = -p \nabla \cdot \mathbf{u}. \quad (198)$$

From conservation of momentum we also have

$$\frac{D}{Dt} \frac{\rho}{2} u^2 = -\mathbf{u} \cdot \nabla p. \quad (199)$$

Combining these two equations we have

$$\frac{\partial E}{\partial t} + \nabla \cdot (E + p) \mathbf{u} = 0, \quad E = \rho \left(e + \frac{1}{2} u^2 \right). \quad (200)$$

At the shock we must therefore require continuity of $(\rho u(e + \frac{p}{\rho} + \frac{1}{2}u^2))$, and since ρu is continuous we have that $e + \frac{p}{\rho} + \frac{1}{2}u^2 = h + \frac{1}{2}u^2$ is continuous:

$$\text{Energy : } h_1 + \frac{1}{2}u_1^2 = h_2 + \frac{1}{2}u_2^2. \quad (201)$$

Let us write $m = \rho u$ as the constant mass flux, and let $v = 1/\rho$. Then conservation of energy may be rewritten

$$h_2 - h_1 = \frac{1}{2}m^2(v_1^2 - v_2^2). \quad (202)$$

Also conservation of momentum can be rewritten

$$m^2 = \frac{p_1 - p_2}{v_2 - v_1} \quad (203)$$

Thus

$$h_1 - h_2 = \frac{1}{2}(v_1 + v_2)m^2(v_2 - v_1), \quad (204)$$

which is equivalent to

$$h_1 - h_2 = \frac{1}{2}(v_1 + v_2)(p_1 - p_2). \quad (205)$$

Written out, this means

$$e_1 - e_2 + p_1v_1 - p_2v_2 = \frac{1}{2}(v_1 + v_2)(p_1 - p_2) \quad (206)$$

or

$$e_1 - e_2 = \frac{1}{2}(v_2 - v_1)(p_1 + p_2). \quad (207)$$

The relations (205),(207) involving the values of the primitive thermodynamic quantities on either side of the shock are called the *Rankine-Hugoniot* relations.

For a polytropic gas we have

$$e = \frac{1}{\gamma - 1}pv. \quad (208)$$

This allows us to write (207) as

$$\frac{2}{\gamma - 1}(p_1v_1 - p_2v_2) + (v_1 - v_2)(p_1 + p_2) = 0, \quad (209)$$

or

$$\frac{\gamma + 1}{\gamma - 1}(p_1v_1 - p_2v_2) - v_2p_1 + p_2v_1 = 0. \quad (210)$$

We now introduce notation from Courant and Friedrichs: Let $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$. (μ has no relation whatsoever to viscosity.) According to (210), if the state p_1, v_1 exists upstream of a shock, the possible downstream states p, v satisfy

$$-p_1v_1 + pv + \mu^2vp_1 - \mu^2pv_1 = 0, \quad (211)$$

or

$$(p + \mu^2 p_1)(v - \mu^2 v_1) + (\mu^4 - 1)p_1 v_1 = 0. \quad (212)$$

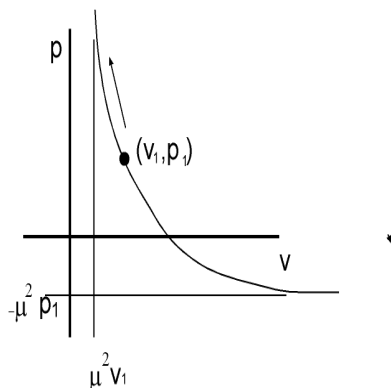


Figure 9. The Hugoniot of the stationary normal shock wave.

We shall later see that the only allowed transition states are upward along the Hugoniot from the point (v_1, p_1) , as indicated by the arrow, corresponding to an increase of entropy across the shock.

8.3.1 Prandtl's relation

For a polytropic gas the energy conservation may use

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{1 - \mu^2}{2\mu^2} c^2. \quad (213)$$

Then conservation of energy across a shock becomes

$$(1 - \mu^2)c_1^2 + \mu^2 u_1^2 = (1 - \mu^2)c_2^2 + \mu^2 u_2^2 \equiv c_*^2. \quad (214)$$

Note that the constancy of $(1 - \mu^2)c^2 + \mu^2 u^2$ implies that $(1 - \mu^2)(u^2 - c^2) = u^2 - c_*^2$. Since $\mu < 1$, this last relation shows that $u > c_*$ iff $u > c$ and $u < c_*$ iff $u < c$.

Prandtl's relation asserts that

$$u_1 u_2 = c_*^2. \quad (215)$$

This implies that on one side of the shock $u > c_*$ and hence $u > c$, i.e. the flow is supersonic relative to the shock position, and on the other side it is subsonic. Since density increases as u decreases, the direction of transition on the Hugoniot indicates that the transition must be from supersonic to subsonic as the shock is crossed.

To prove Prandtl's relation, note that $(1 + \mu^2)p = \rho(1 - \mu^2)c^2$ since $\mu^2 = \frac{\gamma - 1}{\gamma + 1}$. Then, of $P = \rho u^2 + p$,

$$\mu^2 P + p_1 = \mu^2 u_1^2 \rho_1 + (1 + \mu^2) p_1 = \rho_1 [\mu^2 u_1^2 + (1 - \mu^2) c_1^2] = \rho_1 c_*^2. \quad (216)$$

Similarly $\mu^2 P + p_2 = c_*^2 \rho_2$. Thus

$$p_1 - p_2 = c_*^2 (\rho_1 - \rho_2),$$

or

$$c_*^2 = \frac{p_1 - p_2}{\rho_1 - \rho_2} = \frac{p_1 - p_2}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} \frac{1}{\rho_2 \rho_1} = \frac{m^2}{\rho_1 \rho_2} = u_1 u_2, \quad (217)$$

where we have used (203).

8.3.2 An example of shock fitting: the piston problem

Suppose that a piston is driven through a tube containing polytropic gas at a velocity u_p . We seek to see under what conditions a shock will be formed. Let the shock speed be U . In going to a moving shock our relations for the stationary shock remain valid provided that $u - U$ replaces u . Thus Prandtl's relation becomes

$$(u_1 - U)(u_2 - U) = c_*^2 = \mu^2 (u_1 - U)^2 + (1 - \mu^2) c_1^2, \quad (218)$$

where the gas velocities are relative to the laboratory, not the shock. Rearranging, we have

$$(1 - \mu^2)(u_1 - U)^2 + (u_1 - U)(u_2 - u_1) = (1 - \mu^2) c_1^2. \quad (219)$$

Consider now the flow as shown in figure 10.

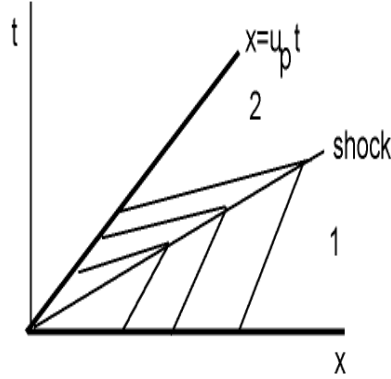


Figure 10. Shock fitting in the piston problem.

The gas ahead of the shock is at rest, $u_1 = 0$, with ambient sound speed $c_1 = c_0$. Behind the shock the gas moves with the piston speed, $u_2 = u_p$. Thus we have a quadratic for U :

$$(1 - \mu^2)U^2 - u_p U = (1 - \mu^2) c_0^2, \quad (220)$$

giving

$$U = \frac{u_p + \sqrt{u_p^2 + 4(1 - \mu^2)c_0^2}}{2(1 - \mu^2)}. \quad (221)$$

We see that a shock forms for any piston speed. If u_p is small compared to c_0 , the shock speed is approximately c_0 , but slightly faster, as we expect. To get the density ρ_p behind the shock in terms of that ρ_0 of the ambient air, we note that mass conservation gives $(u_p - U)\rho_p = -U\rho_0$ or

$$\rho_p = \frac{U}{U - u_p}\rho_0. \quad (222)$$