Stochastic Calculus Fall 2009: Homework 5

1. **Mean reversion**. Consider the daily time-series of dividend/split-adjusted closing prices for Exxon Mobile (XOM), and Chevron Texaco (CVX), from Yahoo!Finance. The respective prices are denoted by X_t and Y_t. We consider the period December 1, 2008 to December 1, 2009. Let $Z_t = log(X_t) - log(Y_t)$. Model this series as an AR-1 process of the form

$$Z_{n+1} = a + bZ_n + c\nu_{n+1},$$

where ν_n are i.i.d. $N(0,1)$ and estimate a, b, c. Can you conclude that the process Z is mean-reverting? [Hint: examine the magnitude of b]. Can you estimate an Ornstein-Uhlembeck process in continuous time that fits the data?

2. **Leveraged and reverse ETFs.** Let X_t and Y_t be Itô processes satisfying the stochastic differential equations

$$\frac{dX_t}{X_t} = \sigma_t dW_t + \mu_t \quad (1)$$

$$\frac{dY_t}{Y_t} = -\sigma_t dW_t - \mu_t = -\frac{dX_t}{X_t} \quad (2)$$

where W_t is a Brownian motion and σ_t and μ_t are non-anticipative with respect to W_t.

Show that

$$dlnX_t + dlnY_t = -\sigma_t^2 dt \quad (3)$$

so that

$$\frac{X_t}{X_0} = \frac{Y_0}{Y_t} \; e^{-\int_0^t \sigma_s^2 ds}. \quad (4)$$

As an application, consider the leveraged funds (leveraged ETFs) with ticker symbols UYG and SKF and denote the (mid-market) prices of these products by X_t and Y_t respectively. The UYG fund provides to investors twice the daily exposure of a financial index tracked by the IShares Dow Jones Financial ETF, which has ticker symbol IYF. On the other hand, SKF provides *minus 2* times the daily variation of IYF. Argue that

$$\frac{dX_t}{X_t} = -\frac{dY_t}{Y_t}$$

is an appropriate model for the relation between the returns of UYG and SKF over short periods of time, comparable to 1 day. Assuming that X_t follows an Itô process, conclude the equation (4) should hold, where σ_s^2 is a trailing estimator of the variance of UYG returns (*e.g.* a 10-day estimator). Verify that this equation holds using December 1, 2008 as the starting point: on each day beginning on December 1 2008 ($t = 0$) until December 1 2009, compare the right and left-hand sides of (4) and show that they agree to within reasonable error.

3. **Drift becomes discounting.** Consider the Cauchy (final-value problem) for the Ornstein-Uhlembeck operator

$$
\begin{aligned}
\frac{\partial\phi(x,t)}{\partial t} - \kappa x \frac{\partial \phi(x,t)}{\partial x} + \frac{1}{2}\frac{\partial^2\phi(x,t)}{\partial x^2} &= 0, \quad (t < T) \\
\phi(x, t = T) &= F(x) \qquad (5)
\end{aligned}
$$

Using Girsanov's theorem, show that

$$
\phi(x,t) = E\left\{ F(W_T)\; e^{-\kappa \int_t^T W_s dW_s - \frac{\kappa^2}{2}\int_t^T W_s^2 ds} |W_t = x \right\}, \qquad (6)
$$

where W_t is Brownian motion. In particular, show that this implies that $\phi(x,t)$ also satisfies:

$$
\phi(x,t) = e^{\frac{\kappa}{2}(x^2 - t)}\; E\left\{ F(W_T)\, e^{-\frac{\kappa}{2}(W_T^2 - T)}\; e^{-\frac{\kappa^2}{2}\int_t^T W_s^2 ds} |W_t = x \right\}. \qquad (7)
$$

Now, use the Feynman-Kac formula and (7) to conclude that $\psi(x,t) = \phi(x,t)e^{-\frac{\kappa}{2}(x^2-t)}$ satisfies the boundary-value problem

$$
\begin{aligned}
\frac{\partial\psi(x,t)}{\partial t} + \frac{1}{2}\frac{\partial^2\psi(x,t)}{\partial x^2} - \frac{\kappa^2}{2}x^2\psi(x,t) &= 0, \quad (t < T) \\
\psi(x, t = T) &= e^{-\frac{\kappa}{2}(x^2 - T)}F(x). \qquad (8)
\end{aligned}
$$

Verify this result directly (without Girsanov's theorem) [by simply using the "change of function" $\psi(x,t) = \phi(x,t)e^{-\frac{\kappa}{2}(x^2-t)}$ and boundary-value problem (5)].

4. **Monte Carlo (again).** Suppose that you want to calculate numerically the probability that the OU process

$$
dX = -\kappa X dt + dW
$$

starts at zero and goes above an extreme level $x = a > 0$ before time T. This corresponds on the PDE side to solving the BV problem (5) with

$$
\begin{aligned}
F(x) &= 0 \;\; if \;\; x < a \\
&= 1 \;\; if \;\; x \geq a \qquad (9)
\end{aligned}
$$

Since the standard deviation of X_t for large values of t is $\frac{1}{\sqrt{2\kappa}}$ (check your notes), the probability will be extremely small if, say, $a = \frac{5}{\sqrt{2\kappa}}$ (a five standard-deviations move for a Gaussian). Here is where formulation (7) can be useful. Using Monte-Carlo simulation and (7), compute the the first-passage probability

$$
P\left\{ \max_{t<1} X_t > 1 | X_0 = 0 \right\}
$$

assuming that $\kappa = 12.5$. Justify your answer...