Stochastic Calculus Fall 2009
Homework 2

1. Conditional Expectation Let (X, Y) be two random variables such that

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{\pi} \quad \text{if } x^2+y^2 \le 1 \\ &= 0 \quad \text{if } x^2+y^2 > 1 \end{aligned} \tag{1}$$

Compute $E(X|Y=y)$, $E(X^2+Y^2|Y=y)$, $E(X+Y|Y=y)$ and the probability density of X. Are X and Y independent?

2. Brownian Scale Invariance Let λ be a positive number and let $X(t)$ be a Brownian path for $0 < t < T$. Show, from the definition of Brownian motion, that $Y(t) = \frac{1}{\lambda}X(\lambda^2 t)$ is distributed like a Brownian motion on the interval $[0, \frac{T}{\lambda^2}]$. Give an interpretation of this result.

3. AR processes (i) Let X_n be a discrete time AR(1) process, such that

$$X_{n+1} = \frac{1}{2}X_n + \frac{1}{8}\nu_{n+1}$$

where ν_k are independent, identically distributed $N(0,1)$. Show that X_n has a long-term, or equilibrium distribution for $n \to \infty$. Also, compute the correlation between X_n and X_{m+n} as $n \to \infty$. (ii) Consider the AR(2) process

$$X_{n+1} = \frac{1}{2}X_n + \frac{1}{3}X_{n-1} + \frac{1}{8}\nu_{n+1}$$

Does X_n admit an equilibrium distribution for $n \gg 1$? If so, compute it. [Hint: write the AR(2) equation as a "vector AR(1)" equation introducing the vector variable $Y_n = (X_n, X_{n-1})$. Solve the latter equation iteratively.]

4. Continuity of paths Taking inspiration from the previous problem, consider a Gaussian process in continuous time such that

$$E(X(t)) = 0, \quad E(X(t+\Delta t)X(t)) = \sigma^2 e^{-\kappa \Delta t}$$

where $\kappa > 0$. This process corresponds to the "continuous time version" of the AR(1) processes discussed in class and in the above problem. Assume that such covariance function is compatible with some stochastic process (something that we shall not prove here). Using Kolmogorov's test, determine whether $X(t)$ can be realized as a process with continuous paths.

5. Wiener's construction of Brownian motion

Consider the following family of functions

$$X_N(t) = \frac{\nu_0 t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_0^N \frac{1}{n} \left(\nu_n(1 - cosnt) + \nu_n' sinnt\right) \tag{2}$$

where ν_k, ν_k' are i.i.d. N(0,1). Show that $X_N(t)$ is a Gaussian process and calculate $E(X_N(t))$, $E(X_N(t)X_N(s))$. Tak the limit as $N \to \infty$ of the covariance and argue, heuristically, that

$$X(t) = \frac{\nu_0 t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_0^\infty \frac{1}{n} \left(\nu_n(1 - cosnt) + \nu_n' sinnt\right) \tag{3}$$

represents a Brownian path. This construction was proposed by N. Wiener in 1923.