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A market-induced mechanism for stock pinning

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Abstract

We propose a model to describe stock pinning on option expiration dates. We argue that if the open interest on a particular contract is unusually large, delta-hedging in aggregate by floor market-makers can impact the stock price and drive it to the strike price of the option. We derive a stochastic differential equation for the stock price which has a singular drift that accounts for the price-impact of delta-hedging. According to this model, the stock price has a finite probability of pinning at a strike. We calculate analytically and numerically this probability in terms of the volatility of the stock, the time-to-maturity, the open interest for the option under consideration and a 'price elasticity' constant that models price impact.

1. Introduction

This paper analyses a phenomenon observed in equity options markets known as 'stock pinning'. Only minutes before options expire, many stock prices are near or at option strike prices. For some stocks, the subsequent evolution of the price until expiration is remarkably different from a random walk. Stock prices will experience a sudden rush to the vicinity of the strike, coupled with the appearance of an unusually high availability of stock offered just above the strike price and similar large size bid just below the strike. Unless important stock-specific news reaches the market, these stocks become *pinned*, i.e. the closing price at expiration will be within a few cents from the strike price.

Historically and, in particular, during the speculative bubble of the late 1990s, traders saw frequent pinning in technology stocks such as Microsoft and Intel. With open interest being very high in several strikes, and with high stock prices and high volatility, several strikes would be 'visited' in a single day. Pinning became apparent only at the very end of the option's lifetime. Krishnan and Nelken (2001) present significant statistical evidence of pinning of Microsoft stock using historical data.

More recently, hedge funds have engaged in trades consisting of selling thousands of put or call options on the same strike in stocks that have normally a much smaller open interest. One example of this activity occurred in the stock J D Edwards (JDEC, option symbol QJD) in 2001. Typical front-month open interests in JDEC are on the order of a few hundred contracts. Nevertheless, over a period of six expirations in 2001, the *same* hedge fund sold repeatedly more than 25 000 contracts on a single strike in the front-month expiration each time. The stock actually pinned at that strike four out of the six times (see figure 1).

It is impossible to determine in advance which stocks may become pinned. Nevertheless, certain conditions can be associated with pinning. Foremost, the *open interest*, which counts the number of outstanding contracts corresponding to a particular strike, is often unusually large. These huge open interests can be as high as 20 or 30 thousand contracts in stocks that average less that 1000 open contracts on any line. In these circumstances, floor market-makers may act as

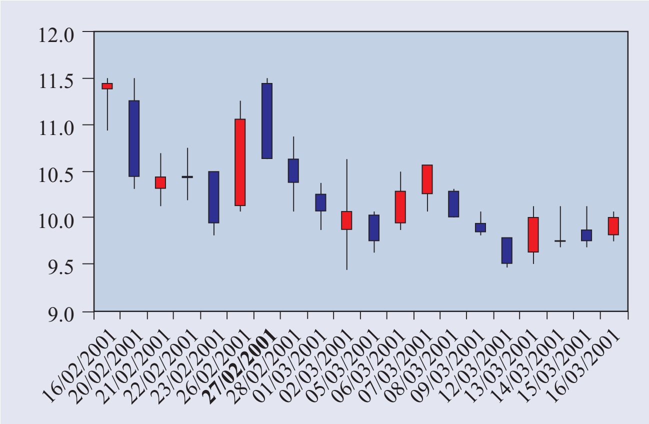

Figure 1. Price evolution of JDEC to March expiration 2001. The open interest on March 10 puts increased from 6125 to 56 128 contracts on 27 February (highlighted). The stock pinned at \$10 on 17 March 2001.

'pinning agents', especially when they are 'long the strike' in aggregate—as in the case of a prior large sale of options by an institution. In this case, since they become 'long gamma', they must hedge their positions by buying stock below the strike and selling stock above the strike, causing pressure on the stock price from above and below. In this paper, we attempt to quantify this phenomenon and to extract some consequences. We propose a model in which market conditions consisting of (a) unusually large open interest and (b) market-makers being long options in aggregate, result in price dynamics whereby pinning occurs with positive, but not certain, probability.

The market conditions that we describe also tend to impact option prices. In particular, they should be accompanied by a drop in the implied volatility of options. In fact, a sale of 25 000 contracts may be impossible to complete all at once, due to the large size of the trade. Market-makers, who act as the buyers, may decide that the risk of owning so many options leads them to decrease their bid price after the initial purchase, and to continue to decrease it until the order is completed. A person observing the market through Black–Scholes glasses would note that the implied volatility decreases. A related effect will be that traders attempting to hedge the trade by selling other strikes will drive down the price of volatility at neighbouring strikes and neighbouring expirations (by trading 'vertical' and/or 'horizontal' spreads).

Recently, Krishnan and Nelken (2001) proposed a model for stock pinning in which the price dynamics is based on a Brownian bridge, i.e. a diffusion which is conditioned to have a specified value at the expiration date. Pinning is modelled by assuming that the (log-)price process behaves like a Brownian bridge with probability p ($p > 0$) and like a standard Brownian motion with probability $1 - p$. The mechanism for pinning is exogenous, in the sense that it is not determined by agents through trading, but rather by introducing a random variable which determines *ex ante* whether the stock will pin or not, regardless of market events taking place between now and the expiration date.

In contrast, the present model links the pinning of the stock to the demand in deltas by market-makers which are long options on a particular strike, assuming that the above conditions hold. This results in a log-price dynamics in which a *force* makes the price drift towards the pinning strike. Using a simple supply/demand argument, we postulate that the force is proportional to the rate of change in the (Black–Scholes) deltas associated with the strike/expiration. The range of this force varies therefore with time, and becomes increasingly concentrated around the strike price as expiration approaches. Consequently, pinning becomes 'endogenous' and is determined by whether or not the log-price becomes 'trapped' in a potential well associated with the force: i.e. on whether the demand/supply of deltas becomes the dominant effect determining the stock price. Even if the pre-conditions stated above (large open interest, market-makers long the strike) hold, the stock may not pin because the force exerted on the price may not be strong enough to drive it to the strike.

To make a fair comparison with Krishnan and Nelken, we note that our model still contains an exogenous parameter, which we call the 'price elasticity of demand' of the stock. However, one advantage of the present model is that pinning is determined self-consistently based on the behaviour of agents in response to stock price changes and the option open-interest. For this reason, we hope that it may have some predictive value and be of practical use by traders.

The rest of the paper is organized as follows: in the next section, we introduce the model as well as the associated dimensionless equations and parameters.

Section 3 discusses the Monte Carlo simulation of the log-price equations and introduces a numerical scheme with adaptive time-mesh that is able to capture pinning events very accurately. The main point is that the force arising from the model is singular near the strike price. In order to obtain numerically accurate simulation results, we must rewrite the equations in such a way that the singularity is eliminated through a change of timescale. We then calculate numerically the pinning probability as a function of the dimensionless variables. We also calculate the cumulative probability distribution of the price near expiration. As expected, the cumulative distribution function exhibits a jump at the strike price, consistent with a positive but not certain probability of pinning.

In section 4, we derive a closed-form solution for the pinning probability, by solving the Fokker–Planck equation with singular drift. As pointed out earlier, however, the 'coupling constant'—which depends on the price elasticity of the stock—is not known in advance, so essentially, the pinning probability cannot be determined solely from observed quantities. In particular, we establish that for a given set of parameter values, there is a finite probability (strictly between zero and one) that pinning takes place.

In section 5, we investigate the effect of the model in terms of option pricing. The main idea is to attempt to measure indirectly the effect of the coupling force on the implied volatilities of traded options, both in the front-month expiration as well as in subsequent expirations. The introduction of an attractive force results in a depletion of implied volatility which is consistent with observations and which might be useful for estimating the 'coupling constant' indirectly or the pinning probability associated with a large sale of options by an institution.

Conclusions are presented in section 6. The appendix contains the more technical mathematical considerations, including rigorous proofs of pinning and estimates for the pinning probabilities.

2. The model

We assume that the trade size impacts stock prices according to the price elasticity equation

$$\frac{\Delta S}{S} = EQ, \tag{1}$$

where Q represents the number of shares traded, S is the stock price, ΔS is the change in stock price associated with a trade of size Q and E is a stock-specific proportionality constant (the price-demand elasticity). Here, Q is a signed quantity, with positive Q representing excess demand and negative Q representing excess supply of stock.

We apply this situation to the case in which the supply/demand for stock is driven by dynamic hedgers having, in aggregate, a long position in n straddles with the same strike price and expiration. Let $\delta(S,\tau)$ represent the delta of a call as a function of the current stock price and time to expiration. The price impact over a small time interval of length Δt caused by the incremental supply/demand for deltas is, from (1),

$$\frac{\Delta S}{S} = -En\frac{\partial \delta(S,\tau)}{\partial t}\Delta t = En\frac{\partial \delta(S,\tau)}{\partial \tau}\Delta t.$$

A first-order approximation for the value of the delta is given by the Black–Scholes formula

$$\delta(S,\tau) = 2N(d_1)$$

with

$$d_1 = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\left(\frac{S}{K}\right) + \left(\mu + \frac{1}{2}\sigma^2\right)\tau\right),$$

where $N(x)$ is the cumulative standard normal distribution function, σ is the implied volatility, S is the spot price, K is the strike price and μ is the rate of carry (interest rate minus dividend rate). Differentiating $\delta(S,\tau)$ with respect to τ, we obtain

$$\frac{\partial \delta(S,\tau)}{\partial \tau} = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{d_1^2}{2}}\left(-\frac{1}{\sigma\tau^{3/2}}\ln\left(\frac{S}{K}\right) + \frac{\mu + \frac{1}{2}\sigma^2}{2\sigma\sqrt{\tau}}\right).$$

This expression can be simplified by introducing the variable

$$y = \ln\left(\frac{S}{K}\right), \tag{2}$$

and setting

$$a = \mu + \tfrac{1}{2}\sigma^2,$$

whereby

$$\frac{\partial \delta(S,\tau)}{\partial \tau} = -\frac{1}{\sqrt{2\pi}}\frac{y - a\tau}{\sigma\tau^{3/2}}\mathrm{e}^{-\frac{(y+a\tau)^2}{2\sigma^2\tau}}.$$

We now derive a stochastic differential equation for y. For this purpose, we assume that the changes in stock price are driven by the price elasticity relation, with an additional noise representing exogenous price fluctuations. Accordingly, the instantaneous returns satisfy the stochastic differential equation

$$\frac{\mathrm{d}S}{S} = nE\frac{\partial \delta(S,\tau)}{\partial \tau}\,\mathrm{d}t + \sigma\,\mathrm{d}W$$

where W is a standard Brownian motion. Applying Ito's formula to expression (2) we conclude that

$$\mathrm{d}y = -\frac{nE}{\sqrt{2\pi}}\frac{y - a\tau}{\sigma\tau^{3/2}}\mathrm{e}^{-\frac{(y+a\tau)^2}{2\sigma^2\tau}}\,\mathrm{d}t + \sigma\,\mathrm{d}W.$$

Introducing the expiration time T, we obtain

$$\mathrm{d}y = -\frac{nE}{\sqrt{2\pi}}\frac{y - a(T-t)}{\sigma(T-t)^{3/2}}\mathrm{e}^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}}\,\mathrm{d}t + \sigma\,\mathrm{d}W. \tag{3}$$

Notice that if $n = 0$, this equation reduces to $\mathrm{d}y = \sigma\,\mathrm{d}W$, which has solution $y = \ln(\frac{S_0}{K}) + \sigma W(t)$. This corresponds to the classical log-normal distribution.

If $n > 0$, the drift term

$$-\frac{nE}{\sqrt{2\pi}}\frac{y - a(T-t)}{\sigma(T-t)^{3/2}}\mathrm{e}^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}}$$

becomes singular as $t \to T$. The effect of this singularity is that the numerator $(y - a(T-t))\mathrm{e}^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}}$ must vanish as we approach the expiration date. In order for this to happen, we must have either $\mathrm{e}^{-\frac{(y+a(T-t))^2}{2\sigma^2(T-t)}} \ll 1$, which means that the log-price process 'escapes' the range of the force through diffusion or, alternatively, that $y - a(T-t) \ll 1$, which is consistent with stock pinning. In the latter case, the price process is 'trapped' near the strike due to supply of stock above the strike price and demand for stock below the strike price.

Before analysing the dynamics of the process further, it is useful to express equation (3) in dimensionless variables. We set

$$z = \frac{y}{\sigma\sqrt{T}}, \qquad s = \frac{t}{T}.$$

Rewriting the equation in these variables, we obtain

$$\mathrm{d}z = -\frac{\beta(z - \alpha(1-s))}{(1-s)^{3/2}}\mathrm{e}^{-\frac{(z+\alpha(1-s))^2}{2(1-s)}}\,\mathrm{d}s + \mathrm{d}\tilde{W}, \qquad 0 < s < 1, \tag{4}$$

where $\tilde{W}$ is a standard Brownian motion and $\alpha = \frac{a\sqrt{T}}{\sigma}$. This shows that there are three parameters that determine the pinning probability and the dynamics:

$$z_0 = \frac{y_0}{\sigma\sqrt{T}} = \frac{1}{\sigma\sqrt{T}}\ln\left(\frac{S_0}{K}\right), \qquad \alpha = \frac{a\sqrt{T}}{\sigma},$$

and

$$\beta = \frac{nE}{\sqrt{2\pi\sigma^2 T}}.$$

The parameter z_0 is the dimensionless logarithmic 'distance' between the price of the stock and the strike, the second

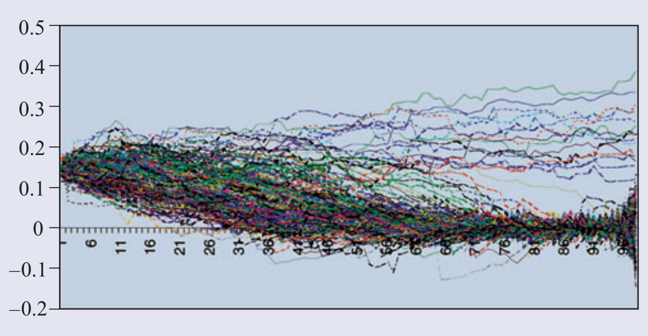

Figure 2. Simulation of the model equations made with a uniform time step. Observe the numerical instability near expiration, which is due to the presence of an unbounded drift. A uniform-Δt scheme is inefficient for calculating pinning probabilities with this model.

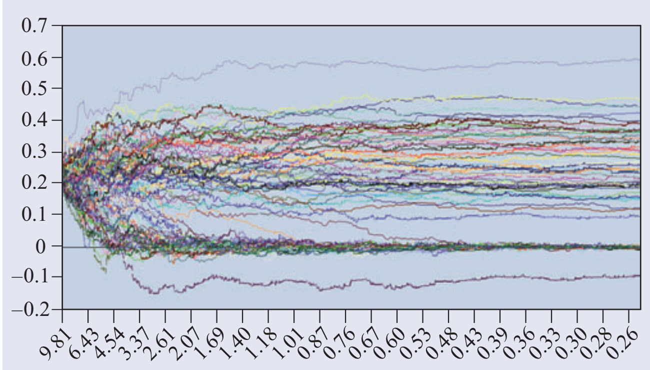

Figure 3. Simulation with uniform-$\Delta\theta$ discretization. The instability is eliminated by refining the timescale as the option expiration approaches. Numbers on the x-axis represent the time-to-maturity measured in days.

parameter depends on the drift assumption made in calculating the theoretical Black–Scholes delta. The parameter β describes the 'strength' of the pinning force. It is proportional to the open interest n and the price elasticity constant, E. It is also inversely proportional to the volatility and to $\sigma\sqrt{T}$, consistent with the fact that increasing the volatility (price impact associated with external information) diminishes the likelihood of pinning and, also, that the probability of pinning is smaller for longer expirations.

3. Simulation of the equations of motion and partial pinning

Simulation of the equations of motion can be done easily using a forward-Euler method in which time is discretized. For simplicity, we will assume that

$$\mu + \tfrac{1}{2}\sigma^2 = 0,$$

i.e. that $a = \alpha = 0$. This simplifies the analysis while retaining the character of the solution.

Since the drift is singular, a uniform mesh may give rise to roundoff errors in the price near the strike and close to expiration (see figure 2 for an illustration of this effect). For this reason, it is convenient to make a change of timescale that eliminates the singularity. We set

$$\frac{\mathrm{d}s}{(1-s)^{3/2}} = \mathrm{d}\theta(s),$$

where θ is the new timescale. Solving for $\theta(s)$, we obtain

$$\theta(s) = \frac{2}{(1-s)^{1/2}} - 2,$$

or

$$s = 1 - \frac{1}{(1+\frac{\theta}{2})^2}.$$

Substituting these expressions into the dimensionless equations of motion (4), we obtain

$$\mathrm{d}z = -\beta z \exp\left\{-\frac{1}{2}\left(1+\frac{\theta}{2}\right)^2 z^2\right\}\mathrm{d}\theta + \frac{1}{\left(1+\frac{\theta}{2}\right)^{3/2}}\,\mathrm{d}\tilde{W}, \qquad 0 < \theta < \infty.$$

The latter equation can be simulated efficiently using a uniform time-step $\Delta\theta$ on a finite but large interval $0 \leqslant \theta \leqslant \theta_{\max}$. In practice, $\theta_{\max}$ is chosen so that if we convert to real time units the simulation stops one or two minutes before expiration. The assessment of pinning is made by defining a small parameter ε and declaring that pinning occurs if[3] $|z_{\theta_{\max}}| < \varepsilon$. The introduction of this time change improves greatly the simulation of paths near expiration, particularly when the stock is near the strike and the drift is strong. In particular, it allows us to estimate the pinning probability accurately. This improvement is displayed in figures 2 and 3 which show ensembles of trajectories generated by uniform-Δt and uniform-$\Delta\theta$ methods.

Simulations were carried out for different values of the dimensionless parameters z_0 and β. In the first set of experiments, we estimated the pinning probabilities for different parameter values, by counting the number of paths that led to $z = 0$ at $s = 1$. We found that, within the range of parameters of interest (which would produce pinning probabilities on the order of 10–30%), convergence to three significant digits was achieved with 4000 paths. The simulations showed that a finite fraction of the paths gives rise to pinning of the stock, this property holding for any positive value of β and all z_0 (see figures 4 and 5). As expected, the fraction of paths for which the stock pins increases with β. For example, at $z_0 = 0$, we find that approximately 18% of the paths will pin for $\beta \approx 0.1$. A set of market parameters consistent with this value would be $nE = 2.85\%$, $\sigma = 40\%$, $T = 30/365$ (30 days until expiration).

We observed also that the fraction of pinned paths decreases as $z_0 > 0$ increases, i.e. as the distance to the strike increases.

Finally, figure 6 shows a graph of the (empirical) cumulative distribution function of the random variable $z(1) = \frac{1}{\sigma\sqrt{T}}\ln(\frac{S(T)}{K})$ for the case $z_0 = 0$, $\beta = 0.1$. The step-like shape

[3] Clearing firms will automatically exercise options that expire in-the-money by 25 cents or more. Therefore, pinning (identifying the final price with the strike) corresponds essentially to a price which is within 25 cents of the strike. This suggests a guideline to define ε for a given stock. In the simulations carried out in this paper, we took a very small value of ε, typically much smaller than the one corresponding to a 25 cent window, in order to approximate the 'true' pinning probability for the continuous-time equation.

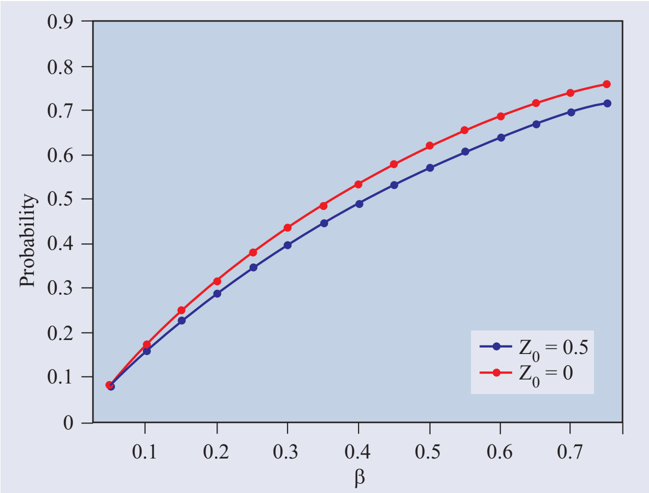

Figure 4. Pinning probability as a function of the parameter β, with $\alpha = 0$. We display results corresponding to two different starting points ($z_0 = 0.0$, $z_0 = 0.5$).

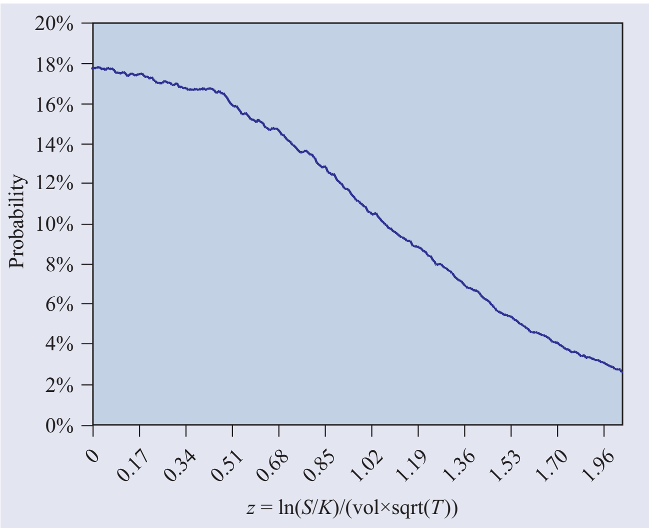

Figure 5. Pinning probability as a function of z_0 computed by Monte Carlo simulation with the adaptive time-step. We use $\beta = 0.1$.

of the graph is consistent with the fact that the distribution has a discrete mass at $z(1) = 0$, i.e. to pinning. The size of the jump corresponds to the pinning probability.

4. A closed-form solution for the pinning probability

In this section, we show that the pinning probability can be computed exactly if $a = \mu + \frac{1}{2}\sigma^2 = 0$. In the appendix, we give an approximate expression for the pinning probability which is valid for all values of a.

We consider the backward Fokker–Planck equation associated with the stochastic differential equation (4). We

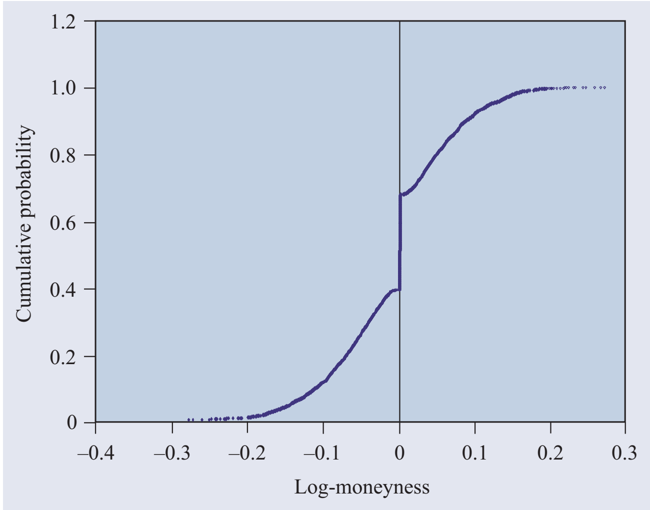

Figure 6. Cumulative probability distribution function computed by Monte Carlo simulation. The step corresponds to the fact that a finite fraction of the paths is pinned at the strike.

Table 1. Pinning probability as a function of z_0 for $\beta = 0.1$.

z_0	p (%)	z_0	p (%)
0.1	17.6	1.1	10.0
0.2	17.4	1.2	9.2
0.3	17.0	1.3	8.0
0.4	16.7	1.4	7.0
0.5	16.5	1.5	6.1
0.6	15.2	1.6	5.2
0.7	14.7	1.7	4.5
0.8	13.6	1.8	3.7
0.9	12.5	1.9	3.3
1.0	11.1	2.0	2.6

assume that $a = \mu + \frac{1}{2}\sigma^2 = 0$. In this case, the equation is

$$\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial z^2} - \frac{\beta z}{\tau^{3/2}} \mathrm{e}^{-\frac{z^2}{2\tau}} \frac{\partial F}{\partial z} = 0, \qquad \tau = 1 - t. \qquad (5)$$

We seek solutions of the form

$$F(z, t) = \mathrm{e}^{\phi\left(\frac{z}{\sqrt{\tau}}\right)/\sqrt{\tau}},$$

where $\phi(\varsigma)$ is an as yet unknown function. Substitution in equation (5) gives rise to the following equation for $\phi(\varsigma)$:

$$\frac{\phi + \varsigma\phi' + \phi''}{2\tau^{3/2}} + \frac{(\phi')^2 - 2\beta\varsigma\phi'\mathrm{e}^{-\frac{\varsigma^2}{2}}}{2\tau^2} = 0.$$

The equation corresponding to the term of order τ^{-2} is the eikonal equation

$$(\phi')^2 - 2\beta\varsigma\phi'\mathrm{e}^{-\frac{\varsigma^2}{2}} = 0,$$

which admits the general solution

$$\phi(\varsigma) = -2\beta\mathrm{e}^{-\frac{\varsigma^2}{2}} + c,$$

where c is an arbitrary constant. Substituting this expression into the term of order $\tau^{-3/2}$ in the Fokker–Planck equation, we find that

$$\phi + \varsigma\phi' + \phi'' = c.$$

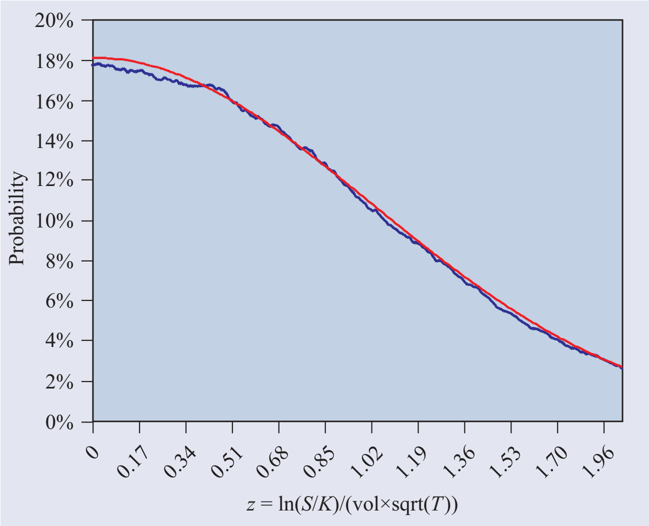

Figure 7. Pinning probability as a function of z_0: Monte Carlo simulations versus exact formula ($\beta = 0.1$).

Therefore, we conclude, setting $c = 0$, that

$$F(z,t) = \exp\left[-\frac{2\beta}{\sqrt{\tau}}e^{-\frac{z^2}{2\tau}}\right] = \exp\left[-\frac{2\beta}{\sqrt{1-t}}e^{-\frac{z^2}{2(1-t)}}\right] \quad (6)$$

is a solution of the Fokker–Planck equation. Set

$$G(z,t) = 1 - F(z,t) = 1 - \exp\left[-\frac{2\beta}{\sqrt{1-t}}e^{-\frac{z^2}{2(1-t)}}\right]. \quad (7)$$

This function satisfies the Fokker–Planck equation with final condition

$$\lim_{t\to 1} G(z,t) = \begin{cases} 1 & \text{if } |z| = 0 \\ 0 & \text{if } |z| \neq 0. \end{cases} \quad (8)$$

From basic principles we know that the probability of pinning for a particle starting at position z at time t,

$$\Pr\{z(1) = 0 | z(t) = z\},$$

satisfies the Fokker–Planck equation in the variables (z, t). Moreover, it satisfies the limiting conditions (8) as $t \to 1$. Therefore, we have

$$\Pr\{z(1) = 0 | z(t) = z\} = G(z,t).$$

We have established the following proposition (see the appendix for details).

Proposition 1. *If* $\mu + \frac{1}{2}\sigma^2 = 0$, *we have*

$$\Pr\{z(1) = 0 | z(0) = z_0\} = 1 - \exp\left[-2\beta e^{-\frac{z_0^2}{2}}\right]. \quad (9)$$

In figure 7 we plot the result of a Monte Carlo simulation of 4000 paths in which the probability of pinning is estimated for different values of z_0 and compare the results with the exact formula. As expected, we obtain nearly perfect agreement between the simulated and exact results.

5. The pricing of options

On the floor of an option exchange such as the AMEX, the response to a very large sell order is immediate. Market-makers reduce the prices of 'nearby' options, i.e. those within the same month on adjacent strikes and those of adjacent months. This occurs in two ways: in the same exchange, as a way of avoiding the purchase of further premium, and also by selling options at reduced prices to market-makers on other exchanges. From the Black–Scholes perspective, the net effect is that implied volatilities decline.

We note also that the appearance of a single, large, sell order is not essential for this effect to occur. As suggested by recent work (Daniels *et al* 2003, Lillo *et al* 2003), breaking a large order into smaller increments is unlikely to produce a gain for the hedge funds. In fact, the price-response to many small orders is likely to be greater than for one large order.

Our model, in which the price dynamics have a singular drift, reflects the volatility contraction by way of a 'lensing effect'. The consequence of having a strike with large open interest is to concentrate a fraction of the paths in a small neighbourhood of the strike (see figure 3). If we calculate option prices as the expected values of cash-flows along such paths, we obtain prices that differ from standard Black–Scholes with volatility σ.

For concreteness, imagine that the stock price is 50 early in month 1, that a large open interest exists on the 50 strike in month 1, and that we are interested in the pricing of options in months 1 and 2. The presence of the 'sticky' strike 50 in month 1 creates the lensing effect; an increased number of paths flow through the vicinity of 50 at the end of the first month. This means that the value of the 50 straddle for month 1 is depressed relative to standard Black–Scholes.

For month 2 there is an equally interesting consequence. Recall that the value of an option is monotone increasing in the volatility parameter, which represents the standard deviation of the logarithmic price movement. In 'path integral' space, this expresses itself by the degree to which paths 'spread out'. Low volatility means a narrower spread of paths out from the source starting point and high volatility the converse. The presence of a 'sticky' strike funnels price paths through the vicinity of the strike. This can also be seen as foreshortening the time axis. Anyone naively inverting the BS pricing formula for options in month 2 would see a reduced implied volatility at all strikes.

6. Conclusions

We proposed a model to explain stock price pinning on option expiration dates. We argue that, under certain circumstances, pinning can be caused by floor traders faced with delta-hedging long-gamma position on a single strike with an unusually high open interest. In such cases, the demand and supply of deltas around the strike can have a significant price impact and may drive the price of the stock to the strike price.

The model consists of an equation for the price dynamics which has a singular drift concentrating near the strike at the expiration date. We show that this model, which depends on a coupling parameter β proportional to the open interest,

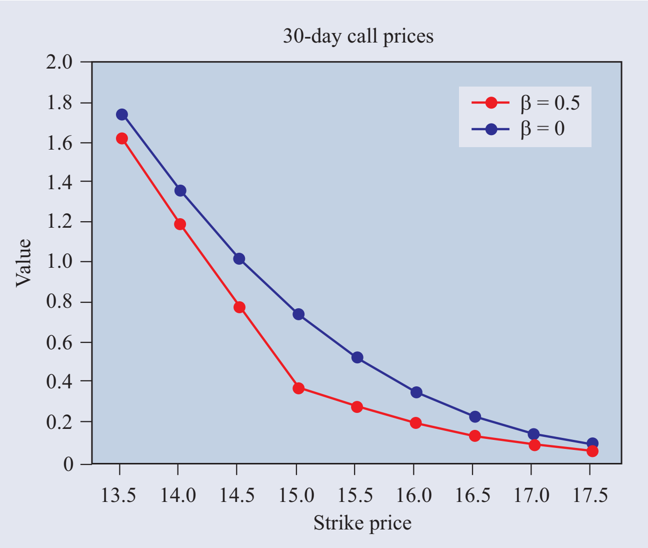

Figure 8. The upper curve represents the Black–Scholes values for call options expiring in 30 days. The lower curve corresponds to the prices of the same options using the model with $\beta = 0.5$. The pinning strike is $K = \$15$.

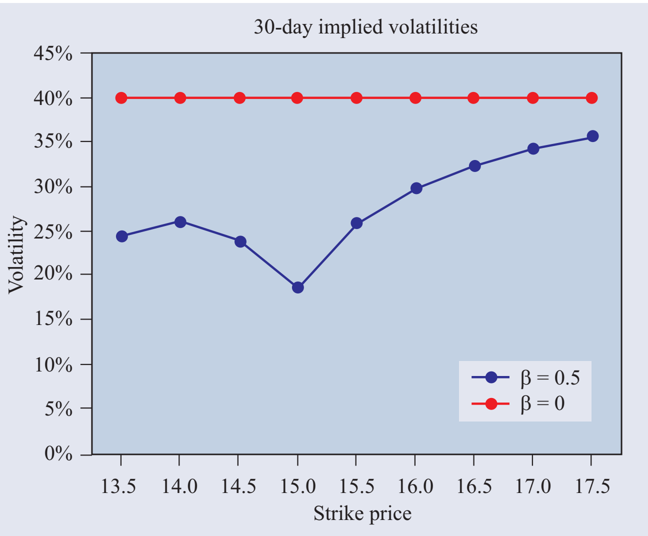

Figure 9. 'Compression' of implied volatilities for front-month options. The parameter values are as in figure 7.

leads to stock pinning with finite probability. Stock pinning is determined by a subtle interaction between the singular drift (associated with price-impact effects), which drives the price to the strike, and independent diffusive shocks which affect the price.

We derived a closed-form formula for the pinning probability in a special case ($\alpha = 0$) and showed rigorously that the pinning effect also holds for all values of α (see appendix). The model predicts quantitatively the impact on option prices due to a trade in which hedge-funds sell large amounts of options to floor market-makers.

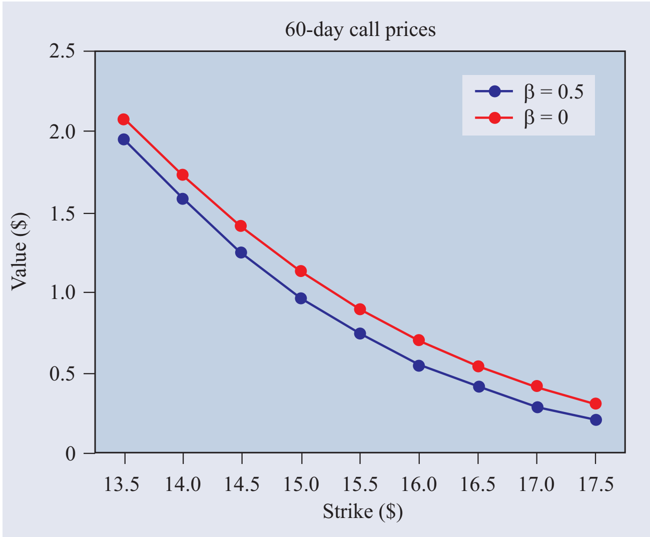

Figure 10. Compression of option prices for options expiring 30 days *after* the expected pinning event. The parameter values are as in figure 7.

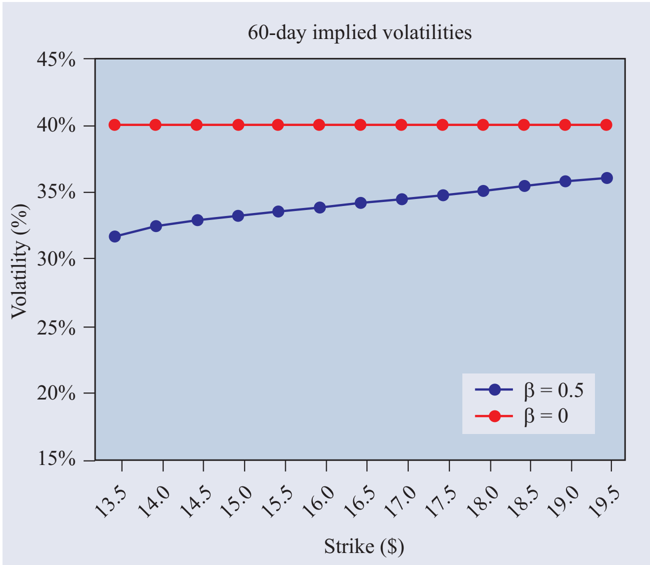

Figure 11. 'Compression' of implied volatilities for options expiring 30 days after the expected pinning event.

Appendix

This appendix provides a rigorous proof of partial pinning (i.e. pinning with positive, but not 100% probability) for all values of the parameter α. First, we establish rigorously proposition 1 of section 4, in which the existence of an exact solution of the Fokker–Planck equation was used to derive a closed-form solution for the probability of pinning for the special case $\alpha = 0$. Secondly, we extend the proof for $\alpha \neq 0$ and derive estimates for the pinning probabilities.

Proof of proposition 1. Let $0 < t^* < 1$, and let $\varepsilon > 0$. We consider the function $G(z, t) = 1 - \exp[-\frac{2\beta}{\sqrt{1-t}} \mathrm{e}^{-\frac{z^2}{2(1-t)}}]$ and the indicator function of the interval $[-\varepsilon, +\varepsilon]$, defined as

$$\chi_\varepsilon(z) = 1 \quad \text{if } |z| \leqslant \varepsilon, \qquad \chi_\varepsilon(z) = 0 \quad \text{if } |z| > \varepsilon.$$

Clearly, for all z, we have

$$\chi_\varepsilon(z) \geqslant G(z, t^*) - G(\varepsilon, t^*). \tag{10}$$

Also, since $F(z, t)$ satisfies the Fokker–Planck equation (5), we have

$$E(G(z(t^*), t^*)|z(t) = z) = G(z, t).$$

Therefore, taking conditional expectations on both sides of (10)

$$\begin{aligned} P(|z(t^*)| \leqslant \varepsilon|z(t) = z) \\ \geqslant E(G(z(t^*), t^*)|z(t) = z) - G(\varepsilon, t^*) \\ = G(z, t) - G(\varepsilon, t^*). \end{aligned}$$

Letting $t^* \to 1$, we conclude that[4]

$$P(|z(1)| \leqslant \varepsilon|z(t) = z) \geqslant G(z, t),$$

whence

$$P(z(1) = 0|z(t) = z) \geqslant G(z, t). \tag{11}$$

This establishes a lower bound on the pinning probability.

To prove the opposite inequality, we use the function

$$F(z, t) = \exp\left[-\frac{2\beta}{\sqrt{1-t}} e^{-\frac{z^2}{2(1-t)}}\right] = 1 - G(z, t).$$

Comparing the graphs of $1 - \chi_\varepsilon(z)$ and $F(z, t^*)$, we note that

$$1 - \chi_\varepsilon(z) \geqslant F(z, t^*) - F(\varepsilon, t^*).$$

Just as before, taking conditional expectations, we obtain

$$\begin{aligned} P(|z(t^*)| > \varepsilon|z(t) = z) \geqslant E(F(z(t^*), t^*)|z(t) = z) \\ - F(\varepsilon, t^*) = F(z, t) - F(\varepsilon, t^*). \end{aligned}$$

Taking the limit $\varepsilon \to 0$ in the latter inequality, we find

$$P(|z(t^*)| > 0|z(t) = z) \geqslant F(z, t) = 1 - G(z, t),$$

and letting $t^* \to 1$,

$$P(|z(1)| > 0|z(t) = z) \geqslant F(z, t) = 1 - G(z, t). \tag{12}$$

This is an upper bound on the pinning probability.

Combining (11) and (12), we conclude that

$$P(z(1) = 0|z(t) = z) = G(z, t),$$

as desired.

The following proposition establishes a quantitative estimate on the pinning probability which implies that partial pinning takes place for all values of $\mu + \frac{\sigma^2}{2} \neq 0$.

[4] The limit of $z(t^*)$ as $t^* \to 1$ is understood in the sense of weak convergence of probabilities (Billingsley 1999).

Proposition 2. *Let* $\alpha = \frac{2\mu+\sigma^2}{2\sigma\sqrt{T}}$ *and let* $z(t)$ *be the solution of the stochastic differential equation (4). For each* δ, $0 < \delta < 112$, *there exists a constant* C *independent of* α, β *and* z *such that*

$$\begin{aligned} &|P(z(1) = 0|z(t) = z) - G(z, t)| \\ &\quad < C\beta^{2\delta}\left(|\alpha| + \frac{\alpha^2}{2}\right)\exp\left(\frac{\alpha^2(1-\delta)}{2\delta}\right)(1-t)^{1/2-\delta}. \end{aligned}$$

Proof. We consider the Fokker–Planck equation associated with the process (4),

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} - \frac{\beta(z - \alpha\tau)}{\tau^{3/2}} e^{-\frac{(z+\alpha\tau)^2}{2\tau}} \frac{\partial f}{\partial z} = 0, \qquad \tau = 1 - t.$$

Notice that the singular drift depends on the parameter α.

Lemma. *Set*

$$L_\alpha f \equiv \frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial z^2} - \frac{\beta(z - \alpha\tau)}{\tau^{3/2}} e^{-\frac{(z+\alpha\tau)^2}{2\tau}} \frac{\partial f}{\partial z},$$

and consider the function $F(z,t) = \exp[-\frac{2\beta}{\sqrt{1-t}} e^{-\frac{z^2}{2(1-t)}}]$. *Then, for every* $\delta \in (0, 1)$, *the following estimate holds:*

$$|L_\alpha F(z, t)| \leqslant \frac{C(\delta, \alpha)\beta^{2\delta}|\alpha|}{(1-t)^{1/2+\delta}}. \tag{13}$$

Here, $C(\delta, \alpha) = C(\delta)(1 + \frac{|\alpha|}{2})\exp(\frac{\alpha^2(1-\delta)}{2\delta})$, *and* $C(\delta)$ *is a numerical constant which depends on* δ *but not on* z, α, β *or* τ.

Accepting this lemma, which will be proved later, and applying Ito's lemma to $G(z(t), t)$, we find that

$$\begin{aligned} &E(G(z(t^*), t^*)|z(t) = z) \\ &= E\left(\int_t^{t^*} L_\alpha G(z(s), s)\, ds|z(t) = z\right) + G(z, t) \\ &= E\left(\int_t^{t^*} L_\alpha F(z(s), s)\, ds|z(t) = z\right) + G(z, t) \\ &\geqslant G(z, t) - \int_t^{t^*} \frac{C(\delta, \alpha)\beta^{2\delta}|\alpha|}{(1-s)^{1/2+\delta}}\, ds. \end{aligned}$$

Taking conditional expectations in inequality (10), we conclude that

$$\begin{aligned} &P(|z(t^*)| \leqslant \varepsilon|z(t) = z) \\ &\geqslant E(G(z(t^*), t^*)|z(t) = z) - G(\varepsilon, t^*) \\ &\geqslant G(z, t) - \int_t^{t^*} \frac{C(\delta, \alpha)\beta^{2\delta}|\alpha|}{(1-s)^{1/2+\delta}}\, ds - G(\varepsilon, t^*) \\ &\geqslant G(z, t) - 4C\beta^{2\delta}|\alpha|((1-t)^{1/2-\delta} - (1-t^*)^{1/2-\delta}) \\ &\quad - G(\varepsilon, t^*). \end{aligned}$$

Letting first $t^* \to 1$ and then $\varepsilon \to 0$, we find that

$$\begin{aligned} &P(|z(1)| = 0|z(t) = z) \\ &\geqslant G(z, t) - 4C(\delta, \alpha)\beta^{2\delta}|\alpha|(1-t)^{1/2-\delta}. \end{aligned}$$

An upper bound for the pinning probability is obtained similarly. This concludes the proof of proposition 2. □

Notice that proposition 2 implies that partial pinning occurs for all values of α, z_0 if $\beta > 0$. In fact, since the drift is non-singular for $t < 1$, there is a positive probability that $z(t)$ be in a given small interval around $z = 0$, $(-q^2, +q^2)$. The time variable t can be chosen sufficiently close to 1 so that $G(z,t) - 4C(\delta,\alpha)\beta^{2\delta}|\alpha|(1-t)^{1/2-\delta} > 0$ for all z in this neighbourhood of zero. It follows that the probability of pinning is finite.

Finally, we establish the lemma.

Proof of the lemma. A straightforward calculation (with $F(z,t) = \exp[-\frac{2\beta}{\sqrt{1-t}}\mathrm{e}^{-\frac{z^2}{2(1-t)}}] = \exp[-\frac{2\beta}{\sqrt{\tau}}\mathrm{e}^{-\frac{z^2}{2\tau}}]$) yields

$$L_\alpha F(z,t) = \frac{2\beta^2}{\tau^2}\exp\left[-\frac{2\beta}{\sqrt{\tau}}\mathrm{e}^{-\frac{z^2}{2\tau}}\right](A_1 + A_2), \tag{14}$$

with

$$A_1 = \frac{z^2}{\tau}\mathrm{e}^{-\frac{z^2}{\tau}}\left(1 - \exp\left(-z\alpha - \frac{\alpha^2\tau}{2}\right)\right)$$

and

$$A_2 = z\alpha\mathrm{e}^{-\frac{z^2}{\tau}}\exp\left(-z\alpha - \frac{\alpha^2\tau}{2}\right).$$

We observe that, for all δ such that $0 < \delta < 1$, we have

$$\exp\left(-z\alpha - \frac{\alpha^2\tau}{2}\right) \leqslant \exp\left(\frac{\delta z^2}{2\tau} + \frac{\alpha^2\tau}{2\delta} - \frac{\alpha^2\tau}{2}\right)$$

and

$$\left|1 - \exp\left(-z\alpha - \frac{\alpha^2\tau}{2}\right)\right| \leqslant \left(|z||\alpha| + \frac{\alpha^2\tau}{2}\right)\exp\left(|z||\alpha| - \frac{\alpha^2\tau}{2}\right) \leqslant \left(|z||\alpha| + \frac{\alpha^2\tau}{2}\right)\exp\left(\frac{\delta z^2}{2\tau} + \frac{\alpha^2\tau}{2\delta} - \frac{\alpha^2\tau}{2}\right).$$

It follows that

$$|A_1| \leqslant \frac{z^2}{\tau}\mathrm{e}^{-\frac{z^2}{\tau}}\left(|z||\alpha| + \frac{\alpha^2\tau}{2}\right)\exp\left(\frac{\delta z^2}{2\tau} + \frac{\alpha^2\tau}{2\delta} - \frac{\alpha^2\tau}{2}\right)$$
$$\leqslant C_1(\delta)\left(|\alpha|\sqrt{\tau} + \frac{\alpha^2\tau}{2}\right)\exp\left(-(1-\delta)\frac{z^2}{\tau}\right) \times \exp\left(\frac{\alpha^2\tau}{2\delta} - \frac{\alpha^2\tau}{2}\right)$$
$$\leqslant C_1(\delta)\left(|\alpha|\sqrt{\tau} + \frac{\alpha^2\tau}{2}\right)\exp\left(-(1-\delta)\frac{z^2}{\tau}\right) \times \exp\left(\frac{\alpha^2\tau(1-\delta)}{2\delta}\right),$$

where $C_1 = C_1(\delta)$ is a constant which is independent of z, α, β and[5] τ. In a similar vein, we have

$$|A_2| \leqslant C_2(\delta)|\alpha|\sqrt{\tau}\exp\left(-(1-\delta)\frac{z^2}{\tau}\right).$$

Substituting these estimates in (14), we find that

$$|L_\alpha F(z,t)| \leqslant C(\delta)\frac{2\beta^2}{\tau^2}\exp\left[-\frac{2\beta}{\sqrt{\tau}}\mathrm{e}^{-\frac{z^2}{2\tau}}\right] \times \left(|\alpha|\sqrt{\tau} + \frac{\alpha^2\tau}{2}\right)\exp\left(-(1-\delta)\frac{z^2}{\tau}\right) \times \exp\left(\frac{\alpha^2(1-\delta)}{2\delta}\right), \tag{15}$$

where $C(\delta)$ is a constant that depends only on δ, but not on z, α, β or τ. Let us find the maximum of the right-hand side of this inequality as a function of z. For this purpose, we introduce the variable $X = \mathrm{e}^{-\frac{z^2}{2\tau}}$ and look for the critical point of the expression

$$X^{2(1-\delta)}\exp\left[-\frac{2\beta}{\sqrt{\tau}}X\right],$$

which is $X^* = \frac{(1-\delta)\sqrt{\tau}}{\beta}$. Substituting this value into the right-hand side of (15), we obtain

$$|L_\alpha F(z,t)| \leqslant C(\delta)\frac{2\beta^2}{\tau^2}\mathrm{e}^{-(2-2\delta)}\left(\frac{1-\delta}{\beta}\right)^{(2-2\delta)} \times \exp\left(\frac{\alpha^2(1-\delta)}{2\delta}\right)\tau^{1-\delta}\left(|\alpha|\sqrt{\tau} + \frac{\alpha^2\tau}{2}\right)$$
$$= C_3(\delta)\exp\left(\frac{\alpha^2(1-\delta)}{2\delta}\right)\beta^{2\delta}\left(|\alpha| + \frac{\alpha^2}{2}\right)\frac{1}{\tau^{1/2+\delta}},$$

which is the desired estimate. The claim is proved, completing the proof of proposition 2. □

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[5] To derive these inequalities we used the fact that if δ and p are positive numbers, then, for all $x > 0$,

$$x^p \leqslant \mathrm{e}^{-p}\left(\frac{p}{\delta}\right)^p \mathrm{e}^{\delta x}.$$

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