Some remarks on VIX futures and ETNs

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Joint work with Andrew Papanicolaou, NYU-Tandon Engineering
The ETF Revolution: From stock baskets to commodities & beyond

- Gold (GLD) physical (storage)
- Crude Oil WTI (USO) synthetic (futures)
- Agriculture (DBA) synthetic (futures)
- Diversified Commoos (DBC) synthetic (futures)
- US Bonds (TLT) physical (T-bonds)
- Currencies (BZF, FXE, FXB, UUP) synthetic (swaps)
- VIX Volatility Index (VXX, VXZ, XIV, UVXY, SVXY) synthetic (futures, notes)
- VSTOXX (EVIX, EXIV) synthetic (futures, notes)
The CBOE S&P500 Implied Volatility Index (VIX)

- Inspired by Variance Swap Volatility (Whaley, 90’s)

\[ \sigma_T^2 = \frac{2e^{\gamma T}}{T} \int_0^\infty OTM(K, T, S) \frac{dK}{K^2} \]

- Here OTM(K, T, S) represents the value of the OTM (forward) option with strike K, or ATM if S=F.

- In 2000, CBOE created a discrete version of the VSV in which the sum replaces the integral and the maturity is 30 days. Since there are no 30 day options, VIX uses first two maturities*

\[ VIX = \sqrt{w_1 \sum_{i=1}^{n} OTM(K_i, T_1, S) \frac{\Delta K}{K_i^2} + w_2 \sum_i OTM(K_i, T_2, S) \frac{\Delta K}{K_i^2}} \]

* My understanding is that recently they could have added more maturities using weekly options as well.
VIX: Jan 1990 to July 2017

Is VIX mean-reverting/stationary?
VIX Descriptive Statistics

<table>
<thead>
<tr>
<th>VIX Descriptive Stats</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>19.51195</td>
</tr>
<tr>
<td>Standard Error</td>
<td>0.094278</td>
</tr>
<tr>
<td>Median</td>
<td>17.63</td>
</tr>
<tr>
<td>Mode</td>
<td>11.57</td>
</tr>
<tr>
<td>Standard Deviation</td>
<td>7.855663</td>
</tr>
<tr>
<td>Sample Variance</td>
<td>61.71144</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>7.699637</td>
</tr>
<tr>
<td>Skewness</td>
<td>2.1027</td>
</tr>
<tr>
<td>Minimum</td>
<td>9.31</td>
</tr>
<tr>
<td>Maximum</td>
<td>80.86</td>
</tr>
</tbody>
</table>

• Definitely heavy tails

• “Vol risk premium theory” implies long-dated futures prices should be above the average VIX.

• This implies that the typical futures curve should be upward sloping (contango) since mode<average
Is VIX a stationary process? Yes and no…

- Augmented Dickey-Fuller test rejects unit root if we consider data since 1990. MATLAB `adftest()`: DFstat=-3.0357; critical value CV= -1.9416; p-value=0.0031.

- Shorter time-windows, which don’t include 2008, do not reject unit root

- Non-parametric approach (2-sample KS test) rejects unit root if 2008 is included.

- We shall assume stationarity and explore some of its consequences in the real-world investing
VIX Futures (symbol: VX)

- Contract notional value = VX × 1,000
- Tick size = 0.05 (USD 50 dollars)
- Settlement price = VIX × 1,000
- Monthly settlements, on Wednesday at 8AM, prior to the 3rd Friday (classical option expiration date)
- Exchange: Chicago Futures Exchange (CBOE)
- Cash-settled (obviously)

Each VIX futures covers 30 days of volatility after the settlement date.
- Settlement dates are 1 month apart.
- Recently, weekly settlements have been added in the first two months.
Settlement dates:

- Sep 20, 2017
- Oct 18, 2017
- Nov 17, 2017
- Dec 19, 2017
- Jan 16, 2018
- Feb 13, 2018
- Mar 20, 2018
- April 17, 2018
Note: Recently introduced weeklies are illiquid and should not be used to build CMF curve.
Partial Backwardation: French election, 1st round
Term-structures before & after French election

Before election (risk-on)

After election (risk-off)
VIX futures: Lehman week, and 2 months later
A paradigm for the VIX futures cycle

• Markets are “quiet”, volatility is low, VIX term structure is in contango (i.e. upward sloping)

• Risk on: the possibility of market becoming more risky arises; 30-day S&P implied vols rise

• VIX spikes, CMF flattens in the front, then curls up, eventually going into backwardation

• Backwardation is usually partial (CMF decreases only for short maturities), but can be total in extreme cases (2008)

• Risk-off: uncertainty resolves itself, CMF drops and steepens

• Most likely state (contango) is restored
Statistics of VIX Futures

- Constant-maturity futures, $V^\tau$, linearly interpolating quoted futures prices

\[
V^\tau_t = \frac{\tau_{k+1} - \tau}{\tau_{k+1} - \tau_k} V X_k(t) + \frac{\tau - \tau_k}{\tau_{k+1} - \tau_k} V X_{k+1}(t)
\]

$V X_k(t)$ = kth futures price on date t, $V X_0$ = VIX, $\tau_0 = 0$, $\tau_k$ = tenor of kth futures
Historical volatility of VX Futures

X-axis: days (0=VIX). Y-axis=daily volatility (annualized)

1 M CMF ~ 65%
5M CMF ~ 35%
PCA: fluctuations from average position

- Select standard tenors $\tau_k$, $k = 0, 30, 60, 90, 120, 150, 180, 210$
- Dates: Feb 8 2011 to Dec 15 2016

\[
\ln V_{t_i}^{\tau_k} = \ln V^{\tau_k} + \sum_{l=1}^{8} a_{il} \Psi_l^{\tau_k}
\]

- Slightly different from Alexander and Korovilas (2010) who did the PCA of 1-day log-returns.

<table>
<thead>
<tr>
<th>Eigenvalue</th>
<th>% variance expl</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>72</td>
</tr>
<tr>
<td>2</td>
<td>18</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5 to 8</td>
<td>&lt;1</td>
</tr>
</tbody>
</table>
Histogram of 1st PCA Weight

Mode is negative
ETFs/ETNs based on futures

- Use futures contracts as a proxy to the commodity itself
- Track an “investable index”, corresponding to a rolling futures strategy
- Fund invests in a basket of futures contracts

\[
\frac{dI}{I} = r \, dt + \sum_{i=1}^{N} a_i \frac{dF_i}{F_i}
\]

\(a_i = \text{fraction (\%) of assets in } ith \text{ future}\)

- Normalization

\[
\sum_{i=1}^{N} a_i = \beta, \quad \beta = \text{leverage coefficient}\]
Average maturity

- Assume $\beta = 1$, let $b_i =$ fraction of total number of contracts invested in $i^{\text{th}}$ futures:

\[
b_i = \frac{n_i}{\sum n_j} = \frac{I a_i}{F_i}.
\]

- The average maturity $\theta$ is defined as

\[
\theta = \sum_{i=1}^{N} b_i (T_i - t) = \sum_{i=1}^{N} b_i \tau_i
\]
Example 1: VXX (maturity = 1M, long futures, daily rolling)

\[
\frac{dI}{I} = rdt + \frac{b(t)dF_1 + (1 - b(t))dF_2}{b(t)F_1 + (1 - b(t))F_2}
\]

Weights are based on 1-M CMF, no leverage

\[
b(t) = \frac{T_2 - t - \theta}{T_2 - T_1}
\]

\[
\theta = 1 \text{ month} = \frac{30}{360}
\]

Notice that since

\[
V_t^\theta = b(t)F_1 + (1 - b(t))F_2
\]

we have

\[
dV_t^\theta = b(t)dF_1 + (1 - b(t))dF_2 + b'(t)F_1 - b'(t)F_2
\]

Hence

\[
\frac{dV_t^\theta}{V_t^\theta} = \frac{b(t)dF_1 + (1 - b(t))dF_2}{b(t)F_1 + (1 - b(t))F_2} + \frac{F_2 - F_1}{b(t)F_1 + (1 - b(t))F_2} \frac{dt}{T_2 - T_1}
\]
Dynamic link between Index and CMF equations (long 1M CMF, daily rolling)

\[
\frac{dl}{l} = r \, dt + \frac{b(t)dF_1 + (1-b(t))dF_2}{b(t)F_1 + (1-b(t))F_2} \\
= r \, dt + \frac{dV_t^\theta}{V_t^\theta} - \frac{F_2 - F_1}{b(t)F_1 + (1-b(t))F_2} \frac{dt}{T_2 - T_1}
\]

Slope of the CMF is the relative drift between index and CMF
Example 2: XIV, Short 1-M rolling futures

This is a fund that follows a DAILY rolling strategy, sells futures, targets 1-month maturity

\[
\frac{dJ}{J} = r \ dt - \frac{dV_t^\theta}{V_t^\theta} + \left. \frac{\partial \ln V_t^\tau}{\partial \tau} \right|_{\tau=\theta} \ d\tau
\]

\( \theta = 1 \text{ month} = 30/360 \)
Stationarity/ergodicity of CMF and consequences

Integrating the $I$-equation for VXX and the corresponding $J$-equation for XIV (inverse):

\[
VXX_0 - e^{-r t} VXX_t = VXX_0 \left[ 1 - \frac{V^\tau_t}{V^\tau_0} \exp \left( - \int_0^t \frac{\partial \ln V^\theta_s}{\partial \tau} ds \right) \right]
\]

\[
e^{-r t} XIV_t - XIV_0 = XIV_0 \left[ \frac{V^\tau_0}{V^\tau_t} \exp \left( \int_0^t \frac{\partial \ln V^\theta_s}{\partial \tau} ds \right) - 1 \right]
\]

Proposition: If VIX is stationary and ergodic, and $E \left( \frac{\partial \ln V^\theta_s}{\partial \tau} \right) > 0$, static buy-and-hold XIV or short-and-hold VXX produce sure profits in the long run, with probability 1.
All data, split adjusted
VXX underwent five 4:1 reverse splits since inception

Flash crash
US Gov downgrade
Huge volume
Taking a closer look, last 2 1/2 years

Note: borrowing costs for VXX are approximately 3% per annum. This means that we still have profitability for shorts after borrowing costs.
Modeling CMF curve dynamics

- VIX ETNs are exposed to (i) volatility of VIX (ii) slope of the CMF curve

- To quantify the profitability of described short VXX/long XIV strategies, we propose a stochastic model and estimate it.

- 1-factor model is not sufficient to capture observed "partial backwardation" and "bursts" of volatility

- Parsimony suggests a 2-factor model

- We build-in mean-reversion to investigate the stationarity assumptions

- Sacrifice other "stylized facts" (fancy vol-of-vol) to obtain analytically tractable formulas
`Classic' log-normal 2-factor model for VIX

\[ \text{\textit{VIX}}_t = \exp(X_{1t} + X_{2t}) \]

\[ dX_1 = \sigma_1 dW_1 + k_1(\mu_1 - X_1)dt \]

\[ dX_2 = \sigma_2 dW_2 + k_2(\mu_2 - X_2)dt \]

\[ dW_1 dW_2 = \rho dt \]

\( X_1 \) = factor driving mostly VIX or short-term futures fluctuations (slow)

\( X_2 \) = factor driving mostly CMF slope fluctuations (fast)

These factors should be positively correlated.
Constant Maturity Futures

\[ V^\tau = E^Q \{VIX^\tau\} = E^Q \{\exp(X_1^\tau + X_2^\tau)\} \]

Ensuring no-arbitrage between Futures, \( Q = \text{"pricing measure" with MPR} \)

\[ V^\tau = V^\infty \exp \left[ e^{-\bar{k}_1 \tau} (X_1 - \bar{\mu}_1) + e^{-\bar{k}_2 \tau} (X_2 - \bar{\mu}_2) - \frac{1}{2} \sum_{j=1}^{2} \sum_{i=1}^{2} \frac{e^{-\bar{k}_i \tau} e^{-\bar{k}_j \tau}}{\bar{k}_i + \bar{k}_j} \sigma_i \sigma_j \rho_{ij} \right] \]

Here, the ‘overline parameters’ correspond to assuming a linear market price of risk, which makes the risk factors \( X \) distributed like OU processes under \( Q \) with renormalized parameters.

Estimating the model consists in finding \( k_1, \mu_1, k_2, \mu_2, \bar{k}_1, \bar{\mu}_1, \bar{k}_2, \bar{\mu}_2, \sigma_1, \sigma_2, \rho, V^\infty \) using historical data.
Stochastic differential equations for ETNs (e.g. VXX)

\[\frac{dI}{I} = r \, dt + \frac{dV_t^\theta}{V_t^\theta} - \left[ \frac{\partial \ln V_t^\tau}{\partial \tau} \right]_{\tau=\theta} \, dt\]

Substituting closed-form solution in the ETN index equation we get:

\[\frac{dI}{I} = r \, dt + \sum_{i=1}^{2} e^{-k_i \theta} \sigma_i dW_i + \sum_{i=1}^{2} e^{-k_i \theta} \left[ k_i (\mu_i - \bar{\mu}_i) + (k_i - \bar{k}_i) (X_i - \mu_i) \right] dt\]

Equilibrium local drift = \[\sum_{i=1}^{2} e^{-k_i \theta} [\bar{k}_i (\mu_i - \bar{\mu}_i)] + r\]

Local variance = \[\sum_{j=1}^{2} e^{-k_j \tau} e^{-k_j \tau} \sigma_i \sigma_j \rho_{ij}\]
Estimating the model, 2011-2016

- Kalman filtering approach

<table>
<thead>
<tr>
<th>Estimated Θ</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Input data: 2/2011 to 12/2016, with VIX and CMFs 1m to 7m.</td>
<td>Input data: 2/2011 to 12/2016, with VIX, 3m and 6m CMFs.</td>
</tr>
<tr>
<td>$\tilde{\mu}_1$</td>
<td>$\tilde{\mu}_1$</td>
</tr>
<tr>
<td>3.8103</td>
<td>3.2957</td>
</tr>
<tr>
<td>$\tilde{\mu}_2$</td>
<td>-0.7212</td>
</tr>
<tr>
<td>1.1933</td>
<td>0.4065</td>
</tr>
<tr>
<td>$\bar{\kappa}_1$</td>
<td>10.8757</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.6776</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.8577</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.4462</td>
</tr>
</tbody>
</table>
Estimating the model, 2007 to 2016 (contains 2008)

<table>
<thead>
<tr>
<th>Estimated Θ</th>
<th>Input data: 7/2007 to 7/2016, with VIX and CMFs 1m to 6m.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\mu}_1$</td>
<td>-6.6216</td>
</tr>
<tr>
<td>$\bar{\mu}_2$</td>
<td>9.7372</td>
</tr>
<tr>
<td>$\bar{\kappa}_1$</td>
<td>0.6543</td>
</tr>
<tr>
<td>$\bar{\kappa}_2$</td>
<td>5.9052</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>0.5525</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.9802</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.6015</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Estimated Θ</th>
<th>Input data: 7/2007 to 7/2016, with VIX, 1m and 6m CMFs.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\mu}_1$</td>
<td>2.4581</td>
</tr>
<tr>
<td>$\bar{\mu}_2$</td>
<td>0.8002</td>
</tr>
<tr>
<td>$\bar{\kappa}_1$</td>
<td>0.5505</td>
</tr>
<tr>
<td>$\bar{\kappa}_2$</td>
<td>10.0013</td>
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<tr>
<td>$\sigma_1$</td>
<td>0.4294</td>
</tr>
<tr>
<td>$\sigma_2$</td>
<td>0.7998</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.5073</td>
</tr>
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</table>
Results of the Numerical Estimation:
Model’s prediction of profitability for short VXX/long XIV, in equilibrium

<table>
<thead>
<tr>
<th></th>
<th>Jul 07 to Jul 16 VIX, CMF 1M to 6M</th>
<th>Jul 07 to Jul 16 VIX, 1M, 6M</th>
<th>Feb 11 to Dec 16 VIX, CMF 1M to 7M</th>
<th>Feb 11 to Jul 16 VIX, 3M, 6M</th>
</tr>
</thead>
<tbody>
<tr>
<td>Excess Return</td>
<td>0.30</td>
<td>0.32</td>
<td>0.56</td>
<td>0.53</td>
</tr>
<tr>
<td>Volatility</td>
<td>1.00</td>
<td>0.65</td>
<td>0.82</td>
<td>0.77</td>
</tr>
<tr>
<td>Sharpe ratio</td>
<td>0.29</td>
<td>0.50</td>
<td>0.68</td>
<td>0.68</td>
</tr>
</tbody>
</table>

Notes:

(1) For shorting VXX one should reduce the ”excess return” by the average borrowing cost which is 3%. It is therefore better to be long XIV (note however that XIV is less liquid, but trading volumes in XIV are increasing.

(2) **Realized Sharpe ratios are higher.** For instance the Sharpe ratio for Short VXX (with 3% borrow) from Feb 11 To May 2017 is 0.90. This can be explained by low realized volatility in VIX and the fact that the model predicts significant fluctuations in P/L over finite time-windows.
Variability of rolling futures strategies predicted by model

Simulated 1 Month ETN

Simulated 1 Month invETN
Thank you!