Stochastic Calculus : Lecture I

We discuss <u>stopping times</u> associated with diffusions & other processes.

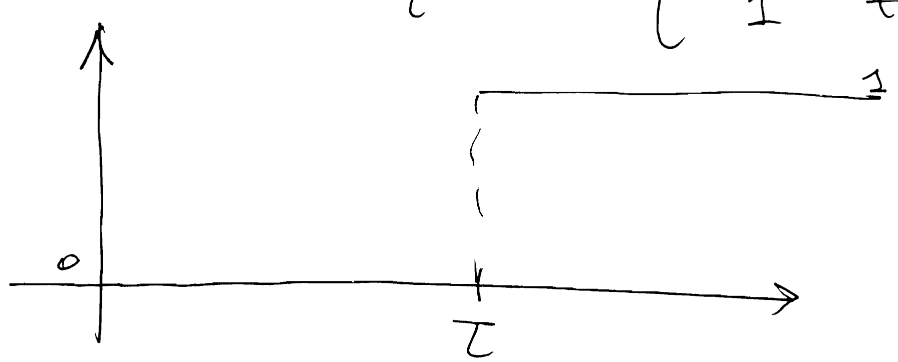
A r.v. Y_t is <u>adapted</u> to $\{X_t\}$ if

$$E\{ Y_t \mid X_s, s \le t \} = Y_t, \qquad a.s.$$

which means that if $\{X_s, s \le t\}$ is ~~known~~ observed, then you know the value of Y_t.

Let τ be a positive r.v. taking values in $(0, \infty)$. Set

$$I_\tau(t) = \begin{cases} 0 & t < \tau \\ 1 & t \ge \tau \end{cases}$$



We say that τ is an X-stopping time if $I_\tau(\cdot)$ is ~~adapted~~ to $\{X\}$ <u>non-anticipatin</u>

Intuitively, this means ~~that~~ the fact that τ is greater or less than t is a function of the path $\{X_s\}_{s \le t}$.

Sometimes, stopping times are called "stopping rules". Think of a stopping time as a "yes/no rule" to act, which is determined by observing the path.

Example 1: First-passage time of BM.

$X_t = W_t$ (B.M. starting at zero)

$a > 0$.

$$\tau_a \equiv \inf\{t : W(t) = a\}.$$

then τ_a is a stopping time, since

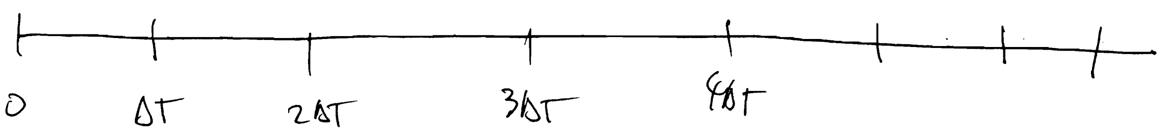
$$\{\tau_a > T\} = \{\sup_{s \le T} W_s < a\}$$

We already showed that

$$P[\tau_a < T] = 2\,P[W(T) > a] = 2\left[1 - N\left(\frac{a}{\sqrt{T}}\right)\right].$$

Example 2: (Killing)

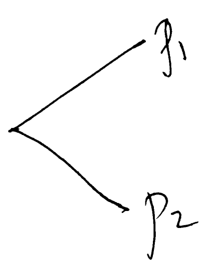
Let X_t be a diffusion process, and $\lambda_t = f(X_t)$ be a positive process. Consider a Poisson process with intensity λ_t.



Define a stopping time as follows

(i) observe the path until time t.

(ii) flip a "coin"



$$p_1 = f(X_t)\,\Delta T \qquad \text{(heads)}$$

$$p_2 = 1 - f(X_t)\,\Delta T \qquad \text{(tail)}$$

(iii) If heads occurs, set $\tau = t + \Delta t$

(iv) If not continue along.

We are adding to X_t a <u>stopping time</u> τ

$$P[\tau > T] = E[P[\tau > T \mid X_s, s \le T]]$$

$$\cong E\left\{ \prod_{i=1}^{n} \left(1 - f(W(t_i))\Delta T\right) \right\}$$

$$= \mathbb{E}\left\{ e^{-\int_0^T f(W(t))ds} \right\}$$

In particular,

$$E\{F(X_T), \tau > T\} =$$

$$= E\{E\{F(X_T); \tau > T \mid X_s, s \le T\}\}$$

$$= E\{F(X_T) P[\tau > T \mid X_s \le s \le T]\}$$

$$= \mathbb{E}\left\{ F(X_T) e^{-\int_0^T f(X_s)ds} \right\}.$$

It follows that if τ is a killing time with intensity $f(X_t)$ then

$$E\left[F(X_T);\ \tau > T \mid X_t = x\right] = \varphi(x,t)$$

where

$$\boxed{\begin{cases} \dfrac{\partial \varphi}{\partial t} + \mathcal{L}\varphi - f\varphi = 0 & 0 \le t \le T \\ \varphi\big|_{t=T} = F(x) & \end{cases}}$$

Computing expectations of diffusions on the set of paths where $\tau < T$ can be done with PDES.

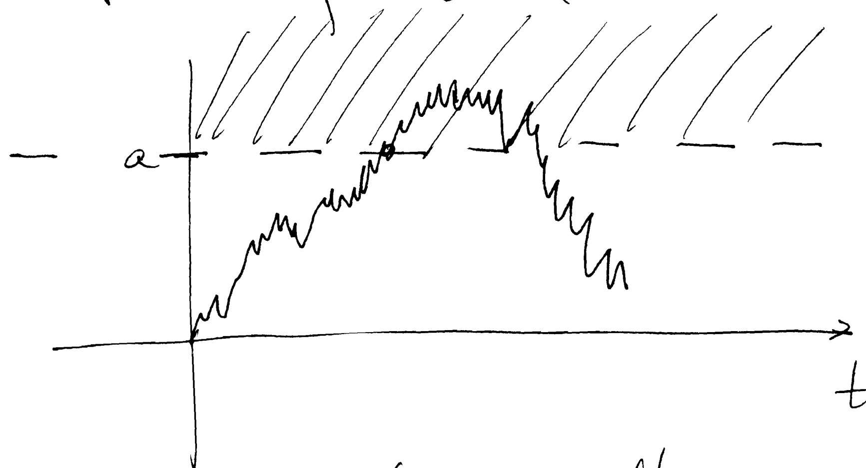
Ex 3: Assuming $X_t = W_t$ and

$$f(x) = \frac{1}{\varepsilon}\, \chi_{[a,+\infty)}(x),$$

define τ_ε as the associated killing time. Notice that

$$\lim_{\varepsilon \downarrow 0} \tau_\varepsilon = \tau_a^*$$

where τ_a^* is the first-passage time of $x=a$. To see this



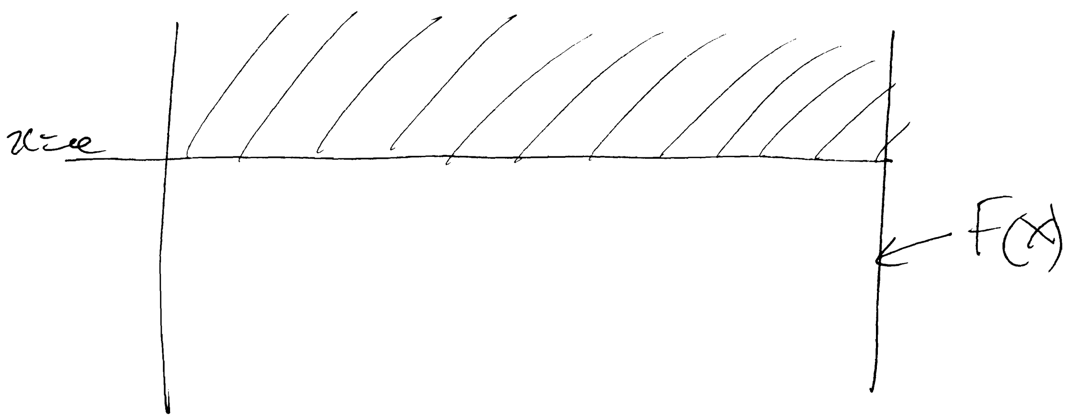
Once the paths crosses over to $\{x > a\}$, the intensity becomes $O(\frac{1}{\varepsilon})$ so the probability of survival for Δt units of time is

$$e^{-\Delta t/\varepsilon}$$

as $\varepsilon \downarrow 0$ the probability is zero.

$$E\left[F(X_T);\ \tau_\varepsilon > T \mid X_t = a\right] = \varphi_\varepsilon(x,t)$$

$$\begin{cases} \dfrac{\partial \varphi_\varepsilon}{\partial t} + \mathcal{L}\varphi_\varepsilon - \dfrac{1}{\varepsilon}\chi_{(a,\infty)}(x)\,\varphi_\varepsilon(x,t) = 0 \\[2ex] \varphi_\varepsilon(x, t=T) = F(x) \end{cases}$$



~~If~~ Clearly $|\varphi(x,t)| < \|F(x)\|_\infty$, so φ is a bounded function. If $x < a$ then the PDE for $\varphi_\varepsilon(x,t)$

$$\begin{cases} \dfrac{\partial \varphi_\varepsilon}{\partial t} + \mathcal{L}\varphi_\varepsilon = 0 & x<a \\ & t<T \\ \varphi_\varepsilon\big|_{t=T} = F(x) \end{cases}$$

Naturally, if $x>a$ $\varphi_\varepsilon(x,t)\downarrow 0$.
This can be seen from the PDE
or directly from

$$\varphi_\varepsilon(x,t)= E\left[F(X_T)\, e^{-\frac{1}{\varepsilon}\int_0^t \chi_{(a,\infty)}(X_s)\,ds} \,\middle|\, X_t=x\right]$$

Since X_t is <u>continuous</u> if $X_t>a$ then $\exists\, \delta t$ $\inf_{s\le\delta t} X_{t+s} > a$,

so passing to the limit give

$$\varphi_\varepsilon(x,t) \longrightarrow 0$$

$\varepsilon\to 0$.

Thus

$$\varphi(x,t)= E\{F(X_T);\ \tau_a^t>T \mid X_t=x\}$$

satisfies the boundary-value

pb.

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0 & x < a \\ \varphi\big|_{t=T} = F(x) \\ \varphi\big|_{x=a} = 0 \end{cases}$$

Cauchy – Dirichlet problem.

Another way to see this result is as follows:

If τ is a stopping time then if g_t is W_t-adapted

$$E\left[\int_t^\tau g_t \cdot dW_t \,\Big|\, t\right] = 0. \quad (\tau > t)$$

Intuitively,

$$\int_t^\tau g_s\, dw_s = \int_t^{\cdot} (1 - I_\tau(s))\, g_s \cdot dw_s .$$

Therefore, if $\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0$,

$$\varphi(X_{\tau\wedge T}, \tau\wedge T) = \varphi(X_t, t) + \int_t^{\tau\wedge T} \varphi_x\, \sigma \cdot dw + \int_t^{\tau\wedge T} \left(\mathcal{L}\varphi + \frac{\partial \varphi}{\partial t}\right) dt$$

$$\varphi(X_{\tau\wedge T}, \tau\wedge T) = \varphi(X_t, t) + \int_t^{\tau\wedge T} \varphi_x\, \sigma\, dw$$

$$E\left[\varphi(X_{\tau\wedge T}, \tau\wedge T) \mid X_s, s\le t; w_s, s\le t\right]$$

$$= \varphi(X_t, t) \qquad \tau > t$$

If

$$E\left[\varphi(X_{\tau\wedge T}, \tau\wedge T) \mid X_t = x\right] = \varphi(x,t)$$

$$= E\left[\varphi(X_T, T);\ \tau > T \mid X_t = x\right] +$$

$$E\left[\varphi(X_\tau, \tau);\ \tau < T \mid X_t = x\right]$$

$$\cancel{= E[F(X_t);\ \tau > T) \mid}$$

$$= E\left[F(X_T);\ \tau > T \mid X_t = x\right] +$$

$$+ E\left[\varphi(a,\tau),\ \tau < T \mid X_t = x\right].$$

$\therefore$ If $\varphi(a,t) \equiv 0$, it follows that

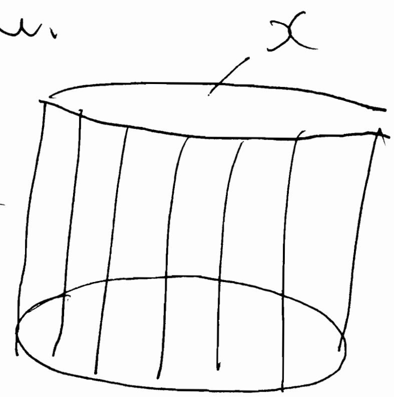
$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0 & x < a,\ t < T \\ \varphi\big|_{t=T} = F(x) & \\ \varphi(x=a) = 0 & \end{cases}$$

represents the conditional expectation that we wish to calculate.

Multidimensional problem

Let $\Omega \in \mathbb{R}^N$ be a domain.

Let τ_Ω be the fpT$_t$ of $\partial\Omega$.



$$E\left[F(X_T);\ \tau_\Omega > T \mid X_t = x\right] = \varphi(x,t)$$

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + \mathcal{L}\varphi = 0 & \text{in } \Omega\times(0,T) \\ \varphi\big|_{t=T} = F(x) & \text{in } \Omega \\ \varphi(x,t)\big|_{x\in\Omega} = 0 & \end{cases}$$

This follows from an applecati of Itô.

Assume that

$$\varphi(x,t) = \begin{cases} F(x) & t = T;\ x \in \Omega \\ g(x,t) & t < T;\ x \in \partial\Omega \end{cases}$$

$$\varphi(x,t) = E\left\{ F\left[X_{\tau_\Omega \wedge T}, T \wedge \tau_\Omega\right] \,\middle|\, X_t = x \right\}$$

$t < T$

Then:

$$\begin{cases} \frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0 & 0 < t < T \\ & x \in \Omega \\ \varphi\big|_{t=T} = F(x) & x \in \Omega \\ \varphi\big|_{x \in \partial\Omega} = g(x,t) & \end{cases}$$

If we take $T \to \infty$ $\quad g = g(x)$

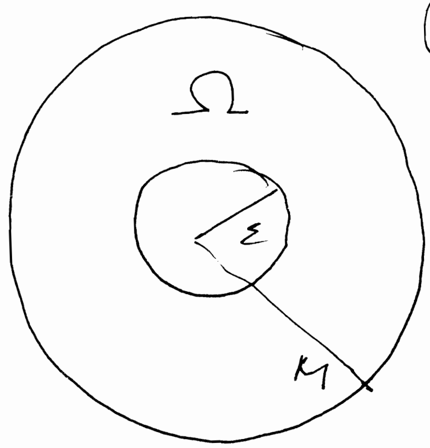
$$\varphi(x) = E\left[g(X_\tau) \mid X_0 = x\right]$$

$$\begin{cases} \mathcal{L}\varphi = 0 & x \in \Omega \\ \varphi\big|_{\partial\Omega} = g(x) & x \in \partial\Omega \end{cases}$$

This is the <u>Dirichlet problem</u>

<u>Example</u>: Find the probability that n-dimensional BM exits $r = M$ before $r = \varepsilon$.

$$\varphi(x) = P\left[X(\tau_\Omega) = M \mid X_0 = x\right]$$



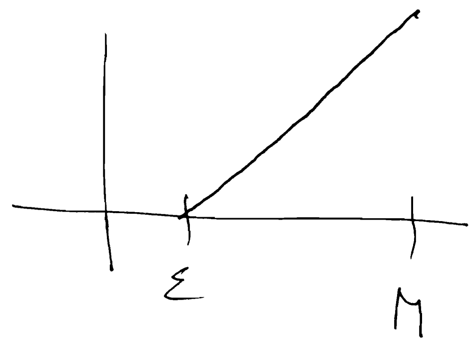
$$\frac{1}{2}\sum_{i=1}^{N} \frac{\partial^2 \varphi}{\partial x_i^2} = 0$$

$$\varphi(x) = 1 \quad |x| = M$$

$$\varphi(x) = 0 \quad |x| = \varepsilon$$

Note: The case $N=1$ is

$$\begin{cases} \varphi''(x) = 0 \\ \varphi(\varepsilon) = 0 \\ \varphi(M) = 1 \end{cases}$$



$$\varphi(x) = \frac{x-\varepsilon}{M-\varepsilon}. \qquad \varepsilon < x < M$$

If $n \geq 2$, then $\varphi = \varphi(r)$

$$\frac{\partial \varphi}{\partial x_i} = \varphi'(r)\frac{\partial r}{\partial x_i}$$

$$= \varphi'(r)\frac{x_i}{r}$$

$$\frac{\partial^2 \varphi}{\partial x_i^2} = \varphi''(r)\frac{x_i^2}{r^2} + \varphi'(r)\left(\frac{1}{r} - \frac{x_i^2}{r^3}\right)$$

$$\sum_i \frac{\partial^2 \varphi}{\partial x_i^2} = \varphi''(r) + \varphi'(r)\frac{N-1}{r}$$

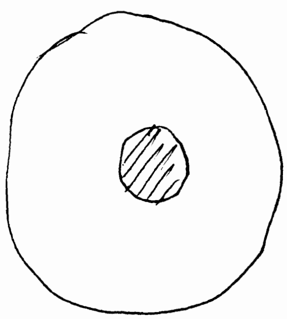
Laplacian for spherical fns.

$$\begin{cases} \varphi''(r) + \dfrac{N-1}{r}\varphi'(r) = 0 & \varepsilon < r < M \\ \varphi(\varepsilon) = 0 & \\ \varphi(M) = 1 & \end{cases}$$

$$\varphi''(r)/\varphi'(r) = -\frac{N-1}{r}$$

$$\ln \varphi'(r) = -(N-1)\ln r + C$$

$$\varphi'(r) = \frac{C}{r^{N-1}} \qquad N \geq 2$$



$$\varphi(r) = \begin{cases} C \ln r + D & N = 2 \\ \dfrac{C}{r^{N-2}} + D & N > 2 \end{cases}$$

$$\varphi(r) = \frac{\ln r - \ln \varepsilon}{\ln M - \ln \varepsilon} \qquad N = 2$$

$$\varphi(r) = \frac{r^{2-N} - \varepsilon^{2-N}}{M^{2-N} - \varepsilon^{2-N}} \qquad N \geq 3$$

If $M \to \infty$ then

$$\lim_{M\to\infty} P[X(\tau)=M] = \begin{cases} 0 & N=2 \\ \dfrac{r^{2-N}-\varepsilon^{2-N}}{-\varepsilon^{2-N}} & N>2 \end{cases}$$

$$\lim_{M\to\infty} P[X(\tau)=M] = \begin{cases} 0 & N=2 \\ 1-\left(\dfrac{\varepsilon}{r}\right)^{N-2} & N\geq 2 \end{cases}$$

$$\text{Prob}\left[\text{hitting } r=\varepsilon \,\middle|\, |X(0)|=r\right] = \left(\frac{\varepsilon}{r}\right)^{N-2}$$

$$\text{Prob.}\left[\lim_{t\to T}|X(t)| = \infty\right] =$$

$$\text{Prob.}\left[\forall M \,;\, \exists T=T(M) \quad |X(t)|>M \quad t>T(M)\right]$$