Stochastic Calculus : Lecture 8

Diffusion Processes $\longleftrightarrow$ Infinitesimal Generator

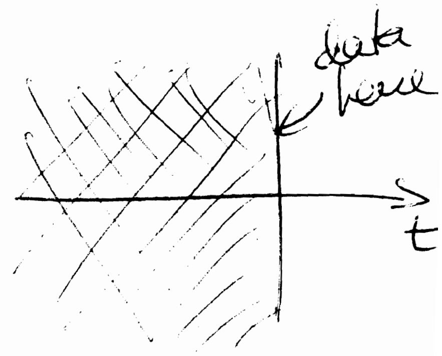
Expectation value $\longleftrightarrow$ Boundary value problem for PDE.

1. $$dX_t = \sigma(X_t, t)\, dZ_t + \mu(X_t, t)\, dt$$

$$\mathcal{L}\phi = \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial x^2} + \mu \frac{\partial \phi}{\partial x}$$

$$E\left\{ F(X_T) \mid X_t = x \right\} = \phi(x, t)$$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \mathcal{L}\phi = 0 & 0 \leq t < T \\ \phi\big|_{t=T} = F(x) & \end{cases}$$



"Cauchy Problem"

$$E\{e^{\lambda W(T)} \mid W(t) = x\} = \varphi(x,t)$$

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 \phi}{\partial x^2} = 0 \\ \phi\big|_{t=T} = e^{\lambda x} \end{cases}$$

Guess: $\phi(x,t) = e^{\lambda x}\,\zeta(t)$

$$\boxed{\dot{\zeta}(t) + \frac{1}{2}\zeta\lambda^2 = 0 \qquad \zeta(T)=1}$$

$$\zeta(t) = c\, e^{-\frac{1}{2}\lambda^2 t}$$

$$\zeta(T) = 1 \implies \zeta(t) = e^{+\frac{1}{2}\lambda^2(T-t)}$$

$$\phi(x,t) = e^{\lambda x + \frac{1}{2}\lambda^2(T-t)}$$

Thus $\phi(0,0) = e^{\frac{1}{2}\lambda^2 T} = E[e^{\lambda W(T)}]$ which we know to be true by Laplace transform.

Introducing a potential, or ~~first~~ zero-order term.

We wish to calculate:

$$\varphi(x,t) \equiv E\left\{ e^{-\int_t^T V(X_s,s)\,ds} F(X(T)) \,\middle|\, X_t = x \right\}$$

Motivation: $V(x,t)$ is a discount rate. For instance, it could represent interest rate in finance.

Consider

$$Y_t = e^{-\int_0^t V(X_s,s)\,ds} \, \Phi(X_t,t)$$

You can think of Y_t as the discounted value at time zero of a cash flow delivered at time t.

By Itô's Lemma

$$dY_t = d\left[e^{-\int_0^t V} \Phi(X_t, t)\right]$$

$$= -VY\,dt + \Phi_t' e^{-\int_0^t V}\,dt$$

$$+ e^{-\int_0^t V}\frac{\partial \Phi}{\partial X}\cdot dX_t +$$

$$+\frac{1}{2} e^{-\int_0^t V}\frac{\partial^2 \Phi}{\partial X^2}\,dX_t^2$$

$$\cancel{= -VY_t\,dt + \Phi}$$

$$= e^{-\int_0^t V(X_s,s)\,ds}\left\{-V\Phi + \Phi_t'\right.$$

$$\left. + \frac{\partial \Phi}{\partial X}\mu + \frac{1}{2}\sigma^2\frac{\partial^2 \Phi}{\partial X^2}\right\}dt$$

$$+ e^{-\int_0^t V}\frac{\partial \Phi}{\partial X}\,\sigma\cdot dZ$$

$$= e^{-\int_0^t V}\left[\frac{\partial\phi}{\partial t} + \mathcal{L}\phi - V\phi\right]dt$$

$$+ e^{-\int_0^t V} \frac{\partial\phi}{\partial x}\sigma \cdot dz$$

Suppose

$$\frac{\partial\phi}{\partial t} + \mathcal{L}\phi - V\phi = 0$$

then:

$$e^{-\int_0^t V(X_s,s)ds} \Phi(X_t,t)$$

is a martingale. It follows that

$$E\left\{e^{-\int_0^T V(X_s,s)ds} \Phi(X_T,T) \mid X_t,t\right\} =$$

$$e^{-\int_0^t V(X_s,s)ds} \Phi(X_t,t)$$

$$E\left\{ e^{-\int_t^T V(X_s, s)\,ds} \Phi(X_T, T) \,\middle|\, X_t,\ X_s\ s \le t \right\}$$

$$= \phi(X_t, t)$$

$$\therefore \boxed{E\left\{ e^{-\int_t^T V(X_s, s)\,ds} \Phi(X_T, T) \,\middle|\, X_t \right\} = \phi(X_t, t).}$$

We conclude that:

If $\phi(x,t)$ satisfies

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \mathcal{L}\phi - V\phi = 0 \quad (t < T) \\ \phi\big|_{t=T} = F \end{cases}$$

Then:

$$E\left\{ e^{-\int_t^T V(X_s, s)\,ds} F(X_T) \,\middle|\, X_t = x \right\} = \varphi(x,t)$$

<u>Example</u> Let X_t be a lognormal process

$$\frac{dX_t}{X_t} = \sigma\, dZ + \mu\, dt$$

$$F(X) = \max(X - K, 0)$$

~~Let~~ Here X models the price of an asset under a suitable probability

σ = volatility

μ = cost of carry

r = interest rate

$K =$ strike price

$T =$ maturity

<u>Problem</u>: Compute

$$E\{e^{-rT}\max\{X_T-K,0\} \mid X_0 = x\}.$$

This represents the theoretical value, or fair value, of an option to buy the asset at time T for K dollars.

PDE formulation: solve

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}x^2\sigma^2 \dfrac{\partial^2 \phi}{\partial x^2} + x\mu \dfrac{\partial \phi}{\partial x} - r\phi = 0 \\ \phi\big|_{t=T} = \max(x-K,0) \end{cases}$$

To solve this, we introduce

$$y = \ln \frac{X}{X_c}$$

$$\begin{cases} \frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial y^2} + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial \phi}{\partial \cancel{x} y} - r\phi = 0 \\ \phi\big|_{t=0} = \max(e^y k, 0) \end{cases}$$

We solve using Fourier i.

$$\begin{cases} \frac{\partial \phi}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial y^2} + \left(\mu - \frac{1}{2}\sigma^2\right)\frac{\partial \phi}{\partial y} - r\phi = 0 \\ \phi\big|_{t=T} = e^{iky} \end{cases}$$

$$\phi = e^{iky}\, \zeta(t,k)$$

$$\begin{cases} \dot{\zeta} - \frac{1}{2}\sigma^2 k^2 \zeta + ik\left(\mu - \frac{1}{2}\sigma^2\right)\zeta - r\zeta = 0 \\ \zeta_{t=T} = 1 \end{cases}$$

$\nu = \mu - \frac{1}{2}\sigma$

$$\zeta(t,k) = e^{-\frac{1}{2}\sigma^2 k^2 (T-t) + ik\nu(T-t) - r(T-t)}$$

$$\boxed{\phi(t,k,y) = e^{iky - \frac{1}{2}\sigma^2 k^2 (T-t) + ik\nu(T-t) - r(T-t)}}$$

It follows that the sol. of the boundary-value pb for the option is

$$\phi(y,t) = e^{-r(T-t)} \int_{-\infty}^{+\infty} (X_0 e^{y} - K)^+ \, e^{-\frac{1}{2\sigma^2 (T-t)}\left[y + \frac{1}{2}\sigma^2 (T-t) + \mu(T-t)\right]^2} \frac{dy}{\sqrt{2\pi (T-t)\sigma^2}}$$

After some "massage", you find the classical BS formula

$$C(X, T, K, r, \mu, \sigma) =$$

$$= e^{-rT}\left[F\,N(d_1) - K\,N(d_2)\right]$$

$$F = X_0\, e^{\mu T} \quad (\text{the } \underline{\text{forward price}})$$

$$\begin{cases} d_1 = \dfrac{1}{\sigma\sqrt{T}} \ln\left(\dfrac{F}{K}\right) + \dfrac{1}{2}\sigma\sqrt{T} \\[2ex] d_2 = \dfrac{1}{\sigma\sqrt{T}} \ln\left(\dfrac{F}{K}\right) - \dfrac{1}{2}\sigma\sqrt{T} \end{cases}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}}$$

<u>Notice</u>: The way we write BS here is more consistent

with financial applications.
We isolate

- the cost of carry of stock
- the cost of money
  interest rate
- the volatility
- the strike/spot
  (moneyness).

This solution technique is known as using the fundamental solution or the Green's function for the problem.
Of course, you need to know it.

Example: $W(t)$ is BM.

Compute:

$$\cancel{\phi(x,t) =} \; \mathbb{E}\left[e^{-\lambda \int_0^T W(s)^2 ds} \right].$$

Solution. Set

$$\varphi(x,t) = \mathbb{E}\left\{ e^{-\lambda \int_t^T W(s)^2 ds} \,\middle|\, W(t) = x \right\}.$$

PDE:

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2} \dfrac{\partial^2 \phi}{\partial x^2} - x^2 \phi = 0 \\ \phi\big|_{t=T} = 1 \end{cases}$$

Guess: $\cancel{\phi(x,t) = e^{-\lambda x \ldots}}$

$$\phi(x,t) = e^{ax^2 + b}$$

$$\begin{cases} \dot{a}x^2 + \dot{b} + \frac{1}{2}\left[2a + 4a^2x^2\right] - \lambda x^2 = 0. \\ a|_{t=T} = 0 \\ b|_{t=T} = 0 \end{cases}$$

$$\begin{cases} \dot{a} + 2a^2 - \lambda = 0 \\ \dot{b} + a = 0 \end{cases}$$

Solution method (Ricatti)

$$\lambda = 2r^2$$

$$\dot{a} = \lambda - 2a^2 = 2(r^2 - a^2)$$

$$\frac{\dot{a}}{r^2 - a^2} = 2$$

$$\frac{1}{2\nu}\left(\frac{\dot{a}}{\nu-a}+\frac{\dot{a}}{\nu+a}\right)=2$$

$$\frac{1}{2\nu}\ \ln\frac{\nu+a}{\nu-a}\Big]_t^T=2(T-t)$$

$$\ln\frac{\nu+a}{\nu-a}\Big]_t^T=4\nu(T-t)$$

$$-\ln\frac{\nu+a(t)}{\nu-a(t)}=4\nu(T-t)$$

$$\ln\frac{\nu+a(t)}{\nu-a(t)}=-4\nu(T-t)$$

$$\frac{\nu+a(t)}{\nu-a(t)}=e^{-4\nu(T-t)}$$

$$\left(1+e^{-4\nu(T-t)}\right)a=\nu\left(e^{-4\nu(T-t)}-1\right)$$

$$\boxed{a(t) = -\nu \cdot \frac{1 - e^{-4\nu(T-t)}}{1 + e^{-4\nu(T-t)}}}$$

The value of $b(t)$ is computed using the second equation

$$\dot{b} + a = 0.$$

$$\phi(x,t) = e^{-\nu\left(\frac{1-e^{-4\nu(T-t)}}{1+e^{-4\nu(T-t)}}\right)x^2 + b(t,T)}$$

$$\phi(0,0) = e^{b(0,T)}.$$

$$b(0,T) = -\nu \int_0^T \frac{1 - e^{-4\nu(T-s)}}{1+e^{-4\nu(T-s)}}\, ds$$

$$\boxed{b = -\nu \int_0^T \frac{1-e^{-4\nu s}}{1+e^{-4\nu s}}\, ds}$$