Stochastic Calculus: Lecture 7

1. Diffusion Processes

A diffusion process is an Itô process for which the local characteristics depend on the current state of the process (as opposed to being general non-anticipative processes)

In equations:

$$dX_t = \sigma(X_t,t)\cdot dZ_t + \mu(X_t,t)\,dt$$

$$\sigma_t = \sigma(X_t,t) \quad , \quad \mu_t = \mu(X_t,t)$$

According to the discussion of last time on Itô processes,

$$E\{\, X_{t+\Delta t} - X_t \mid t \,\} = \mu(X_t,t)\cdot\Delta t + o(\Delta t)$$

$$E\{\, (X_{t+\Delta t} - X_t)^2 \mid t \,\} = \sigma^2(X_t,t)\,\Delta t + o(\Delta t)$$

Diffusion processes are <u>Markov processes</u>:

$$E\{ f(X_t) \mid X_u, u \leq s \} =$$
$$E\{ f(X_t) \mid X_t \}$$

A diffusion process is a stochastic model which is specified by
- the local volatility $\sigma(x,t)$
- the local drift $\mu(x,t)$.

If we apply Itô's Lemma to the diffusion process (1-d)

$$dX_t = \sigma(X_t, t).dz + \mu(X_t, t)dt$$

then $\forall \varphi(x,t)$

$$d\varphi(X_t, t) = \frac{\partial \varphi}{\partial X} . dX_t + \frac{\partial \varphi}{\partial t} . dt + \frac{1}{2}\sigma^2(X_t, t)\frac{\partial^2 \varphi}{\partial X^2}$$
$$= \frac{\partial \varphi}{\partial X}\sigma dW + \frac{\partial \varphi}{\partial t} + \frac{1}{2}\sigma^2(X_t, t)\frac{\partial^2 \varphi}{\partial X^2} + \mu(X_t, t)\frac{\partial \varphi}{\partial x}$$

$$d\varphi(X_t,t) = \frac{\partial\varphi}{\partial x}(X_t,t)\sigma\, dW +$$

$$+\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2\varphi}{\partial x^2} + \mu(x,t)\frac{\partial\varphi}{\partial x}\right)\cdot dt$$

$$= \frac{\partial\varphi}{\partial x}(X_t,t)\,\sigma(X_t,t)\cdot dW_t +$$

$$\left(\frac{\partial\varphi}{\partial t} + \mathcal{L}\right)\varphi(X_t,t)\, dt$$

$$\boxed{\mathcal{L}\varphi(x,t) = \frac{1}{2}\sigma^2(x,t)\frac{\partial^2\varphi}{\partial x^2} + \mu(x,t)\frac{\partial\varphi}{\partial x}(x,t)}$$

$\mathcal{L}$ is a second-order partial differential operator. It is called the <u>infinitesimal generator</u> of the process $\{X_t\}$.

Example:

1. Brownian Motion $\sigma = 1$, $\mu = 0$
   Infinitesimal gen: $\mathcal{L} = \frac{1}{2}\frac{\partial^2}{\partial x^2}$
2. The Ornstein-Uhlembeck Process:

$$dX_t = k(\theta - X_t)dt + \sigma dW_t$$

$$\begin{cases} \mu(x,t) = k(\theta - x) \\ \sigma(x,t) = \sigma \end{cases}$$

$$dX_t + kX_t\,dt = k\theta\,dt + \sigma dW$$

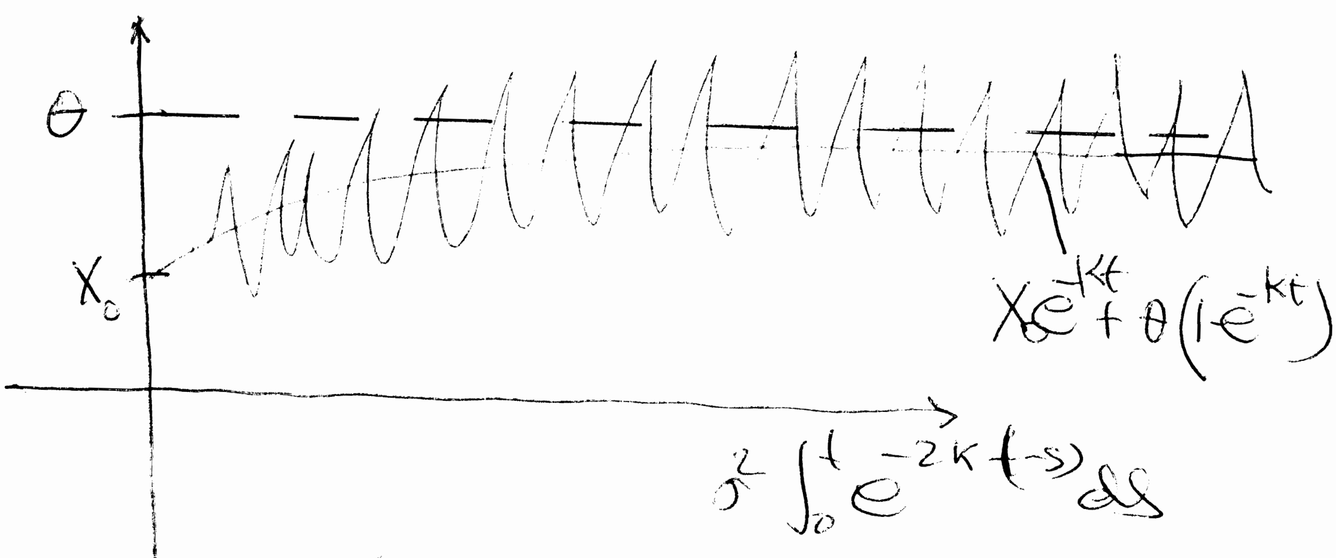
From classical SDE use e^{-kt} as integrating factor,

$$d(e^{kt}X_t) = e^{kt}dX_t + ke^{kt}X_t\,dt$$
$$= e^{kt}(dX_t + kX_t\,dt)$$

$$d\left(e^{kt}X_t\right) = k e^{kt}\theta\, dt + e^{kt}\sigma\, dW_t$$

$$e^{kt}X_t - X_0 = k\,\frac{e^{kt}-1}{k}\,\theta + \int_0^t e^{ks}\sigma\, dW_s$$

$$X_t = e^{-kt}X_0 + \left(1-e^{-kt}\right)\theta + \\ + \int_0^t e^{-k(t-s)}\sigma\, dW_s$$



$$X_t \sim N\left[e^{-kt}X_0 + \left(1-e^{-kt}\right)\theta;\ \sigma^2\left(\frac{1-e^{-2kt}}{2kt}\right)\right]$$

The probability distribution of X_t is Gaussian,

$$E(X_t) = X_0 e^{-kt} + \theta\left(1 - e^{-kt}\right)$$

$$\sigma^2(X_t) = \sigma^2 \frac{1 - e^{-2kt}}{2k}$$

The infinitesimal generator is

$$\mathcal{L}\varphi = \frac{1}{2}\sigma^2 \frac{\partial^2 \varphi}{\partial x^2} + k(\theta - x)\frac{\partial \varphi}{\partial x}$$

3. Multivariate log-normal diffusion

$$\frac{dX^i}{X^i} = \sum_{k=1}^{m} \sigma^i_k \, dZ_k + \mu^i dt$$

$$\boxed{\mathcal{L}\varphi = \frac{1}{2}\sum_{ij=1}^{n} a_{ij} X_i X_j \frac{\partial^2 \varphi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \mu_i X_i \frac{\partial \varphi}{\partial X_i}}$$

$$a_{ij} = \sum_{k=1}^{m} \sigma_{ik} \sigma_{jk}$$

In the case of 1-d diffusion (log normal),

$$\frac{dX}{X} = \sigma dW + \mu dt,$$

$$\boxed{\mathcal{L}\varphi = \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \varphi}{\partial X^2} + \mu X \frac{\partial \varphi}{\partial X}}$$

$$\begin{cases} X \dfrac{\partial \varphi}{\partial X} = \text{change in } \varphi \text{ for a percent change in } X \\ X^2 \dfrac{\partial^2 \varphi}{\partial X^2} = \text{change ie change in } \varphi \text{ for a percent}^2 \text{ change in } X \end{cases}$$

Connection between diffusion processes and PDE.

Let X_t be a 1-dimensional diffusion:

$$dX_t = \sigma(X_t, t)\cdot dZ_t + \mu(X_t, t)\,dt,$$

and let $\varphi(x,t)$ be a smooth function. Then,

$$d\,\varphi(X_t, t) = \sigma(X_t, t)\frac{\partial \varphi}{\partial x}(X_t, t)\cdot dZ_t + \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(X_t, t)\,dt$$

$$\varphi(X_t, t) = \varphi(X_s, s) + \int_s^t \sigma(X_u, u)\frac{\partial \varphi}{\partial x}(X_u, u)\,dZ_u + \int_s^t \left(\frac{\partial \varphi}{\partial u} + \mathcal{L}\varphi\right)(X_u, u)\,du.$$

$$E\{\varphi(X_t,t)\mid s\} =$$

$$= \varphi(X_s,s) + \int_s^t E\left[\left(\frac{\partial\varphi}{\partial u} + \mathcal{L}\varphi\right)(X_u,u)\Big| s\right]du$$

In particular, suppose that

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + \mathcal{L}\varphi = 0 \quad ; t<T, \\ \varphi(x,T) = F(x). \end{cases}$$

~~$E\{F(X_T)\}$~~

$$E\{F(X_T)\mid t\} = \varphi(X_t,t)$$

$$\boxed{E\{F(X_T)\mid X_t\} = \varphi(X_t,t)}.$$

This gives a strong and useful "tool" for computing expected values of diffusion processes.

Example: $X_t = W_t$ = Brownian Motion.

$$E\left[|W_T| \,\middle|\, W_t = x\right] = \varphi(x,t)$$

$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \dfrac{1}{2}\dfrac{\partial^2 \varphi}{\partial x^2} = 0 \\ \varphi(x,T) = |x| . \end{cases}$$

$$\varphi(x,t) = \int_{-\infty}^{+\infty} |y| \, e^{-\frac{(x-y)^2}{2(T-t)}} \frac{dy}{\sqrt{2\pi(T-t)}} ,$$

because $\dfrac{e^{-\frac{(x-y)^2}{2(T-t)}}}{\sqrt{2\pi(T-t)}}$ is the FS of the heat equation.

Boundary Conditions.

$D = \{x : a < x < b\}$

τ_D = first time a path hits the boundary of D

Itô's Lemma is valid between any two deterministic times. It can be generalized to random stopping times.

τ is a stopping time if $\{\tau > t\}$ is measurable with respect to $\{X_u, u \le t\}$.

Intuitively, a stopping time is

a function of the path such that

$$f(t) = \begin{cases} 1 & \tau \leq t \\ 0 & \tau > t \end{cases}$$

is non-anticipative with respect to $\{X_t\}$ or $\{Z_t\}$.

From

$$d\varphi(X_t,t) = \frac{\partial \varphi}{\partial X}(X_t,t)\cdot\sigma(X_t,t)\cdot dZ_t$$

$$+ \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(X_t,t)\,dt$$

we can write

$$\varphi(X_t,t) = \varphi(X_s,s) + \int_s^t \sigma \frac{\partial \varphi}{\partial X}\, dZ \; +$$

$$+ \int_s^t \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi\right)(X_s,s)\,ds \text{, and}$$

if $s > \tau_D$

$$\boxed{\begin{aligned}\varphi(X_{t\wedge\tau_D}, t\wedge\tau_D) = {}& \varphi(X_s, s) + \int_s^{t\wedge\tau_D} \sigma \frac{\partial\varphi}{\partial x} \cdot dz \; + \\ & \int_s^{t\wedge\tau_D} \left(\frac{\partial\varphi}{\partial t} + \mathcal{L}\varphi\right)(X_u, u)\, du \end{aligned}} \quad (**)$$

Furthermore:

$$\mathbb{E}\left[\int_s^{t\wedge\tau_D} \sigma \frac{\partial\varphi}{\partial x} \cdot dz \;\middle|\; s\right] = 0 .$$

This is true because τ_D is a stopping time.

- You can think about this as

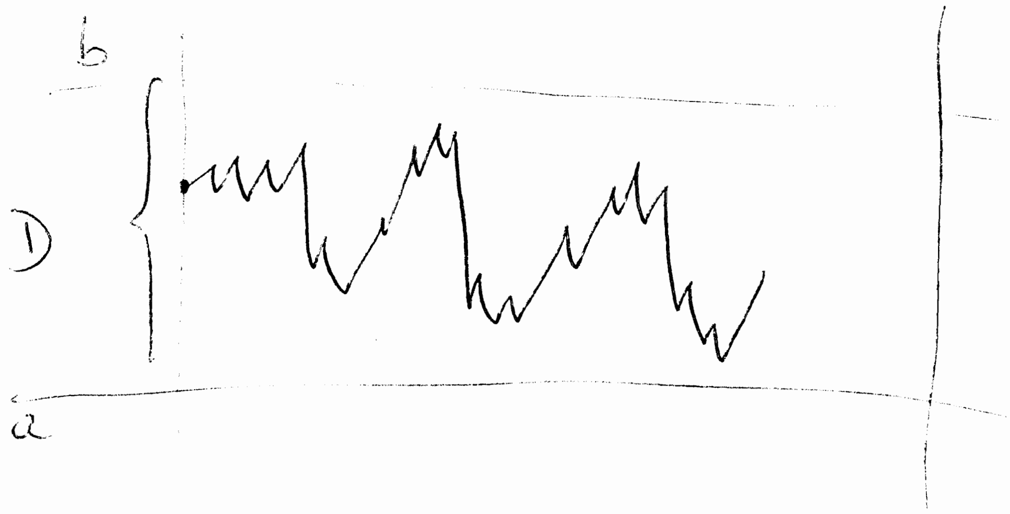
$$\int_s^{t \wedge \tau_D} f \cdot dz = \int_s^t \left(1_{\tau < t} \cdot f \right) \cdot dz.$$

This shows that the left-hand side is a stochastic integral and ~~thus~~ has zero conditional expectation.

Taking conditional expectations in $(**)$ if $\tau_D > t$,

$$E\left\{ \varphi\left(X_{T \wedge \tau_D}, T \wedge \tau_D\right) \middle| X_t \right\} =$$

$$\varphi(X_t, t) + E\left\{ \int_t^{T \wedge \tau_D} \left(\frac{\partial \varphi}{\partial t} + \mathcal{L}\varphi \right) \middle| X_t \right\}$$



If
$$\begin{cases} \dfrac{\partial \varphi}{\partial t} + \mathcal{L}\varphi = 0 & a<x<b, \; t<T \\ \varphi\big|_{t=T} = F(x) \\ \varphi\big|_{\partial D} = 0 \end{cases}$$

then
$$\boxed{\mathbb{E}\left\{ F(X_T);\ \tau_D > T \;\middle|\; X_t \right\} = \varphi(X_t, t)}$$