Stochastic Calculus : Lecture 6

Let $\cancel{X} = (X_1(t), \ldots X_m(t))$ be m independent Brownian motions on $(0,T)$.
Let $\{\sigma_{ik}(t)\}$ $i=1\ldots n$, $k=1,\ldots m$ be non anticipative fns and let $\mu_i(t)$ $i=1\ldots n$ be non-anticipative with respect to $(X_1 \ldots X_m)$.

Consider the process

$$Y_i(t) = Y_i(0) + \int_0^t \sum_k \sigma_{ik}(s)\, dX_k(s) + \int_0^t \mu_i(s)\, ds.$$

or,

$$dY_i(t) = \sum_{k=1}^{m} \sigma_{ik}(t)\, dX_k(t) + \mu_i(t)\, dt$$

$$i = 1 \ldots n$$

This process is known as an Itô process.

Example: Let $\sigma_i(t)>0$ be given functions (processes) and $R_{ij}(t)$ be a process representing the correlation function of the returns of n stocks.

$$E\left[r_i(t)\, r_j(t)\right] = \sigma_i(t)\sigma_j(t)\, R_{ij}(t)\, dt$$

Assume that R_{ij} has m non-zero eigenvectors

$$R_{ij} = \left[V_{ik}\right] \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_m & \\ & & & 0_{\ddots} \end{bmatrix} V^T$$

$$r_i(t) = \sigma_i(t)\left[\sum_{k=1}^{m} \sqrt{\lambda_k}\, V_{ik}(t)\, dX_k(t)\right] + \mu_i(t)\, dt$$

$$\boxed{\frac{dS_i(t)}{S_i(t)} = \sigma_i(t)\left[\sum_{k=1}^{m} \sqrt{\lambda_k}\, V_{ik}(t)\, dX_k(t)\right] + \mu_i(t)\, dt}$$

$$r_i(t) = \frac{S_i(t+\Delta t) - S_i(t)}{S_i(t)}$$

$$\boxed{dS_i(t) = S_i(t)\cdot\sigma_i(t)\left(\sum_{k=1}^{M}\sqrt{\lambda_k}V_{ik}(t)\,dX_k(t)\right) + S_i(t)\,\mu_i(t)\,dt}$$

Thus, Itô processes provide a natural class of models for representing the evolution of a group of stocks, for example.

The parameters $\{\sigma_i(t)\}$, $\{V_{ik}(t)\}$, $\{\lambda_k(t)\}$ $\{\mu_i(t)\}$ can be estimated from data.

Example 2: Stock returns are conditionally normal, with stochastic volatility

$$r_i(t) = \sigma_i(t).\, dX(t)$$
$$\Delta\sigma(t)/\sigma(t) = k(t)\cdot dW(t)$$

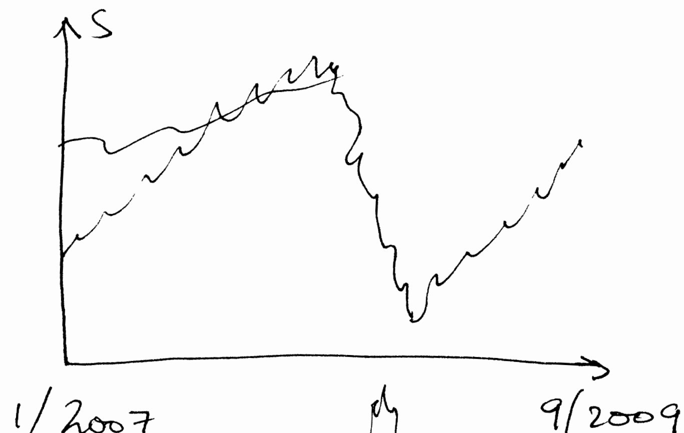
$$\frac{dS_t}{S_t} = \sigma_t \, dX_t$$

$$\frac{d\sigma_t}{\sigma_t} = \kappa_t \, dW_t$$

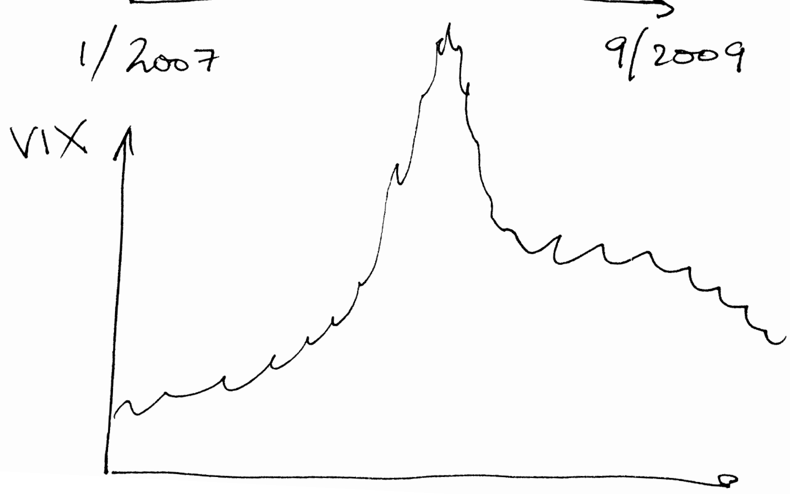
$$E(dX_t \, dW_t) = \rho \, dt$$

<u>Example</u>: Let S_t = S&P 500 index, σ_t = volatility of S&P 500 index

& <u>Data</u>



index



index
implied
volatility

Take vix as proxy for σ_t.

$$\begin{cases} \dfrac{\Delta S_t}{S_t} \simeq N(0, \sigma_t^2) \\[2ex] \dfrac{\Delta \sigma_t^2}{\sigma_t} \simeq N(0, K^2) \end{cases}$$

discrete model

$$\frac{\Delta\sigma}{\sigma} = \beta \frac{\Delta S}{S} + \varepsilon$$

Regression

Links S and σ.

$$\frac{dS}{S} = \sigma \cdot dW_1$$

$$\frac{d\sigma}{\sigma} = \beta \frac{dS}{S} + K_2 \, dW_2$$

$$\begin{cases} \dfrac{dS_t}{S_t} = \sigma_t \, dW_1(t) \\[2ex] \dfrac{d\sigma_t}{\sigma_t} = \beta_t \sigma_t \cdot dW_1(t) + K_2 \cdot dW_2 \end{cases}$$

$$\beta_t = \frac{K\rho}{\sigma}$$

$$\begin{cases} \dfrac{dS_t}{S_t} = \sigma_t \, dW_1(t) \\[2ex] \dfrac{d\sigma_t}{\sigma_t} = K\left(\rho \, dW_1(t) + \sqrt{1-\rho^2} \, dW_2(t)\right) \end{cases}$$

Both the multivariate model for stock prices as the model for a stock price with stochastic volatility are widely used in mathematical finance.

Returning to Itô Processes. Consider a general Itô process

$$dX_i = \sum_{k=1}^{m} \sigma_{ik} \cdot dW_k + \mu_i \, dt$$

- $\{W_k, k=1\dots n\}$ independent Brownian motions.
- σ_{ik} non-anticipative with respect to $\{dW_k\}_{k=1}^{m}$
- μ_i non-anticipative with respect to $\{W_k\}_{k=1}^{m}$

In integral form:

(7)

$$X_i(t) = X_i(0) + \int_0^t \sum_{k=1}^{m} \sigma_{ik} \cdot dW_k(t) + \int_0^t \mu_i(s)\,ds.$$

Itô processes are processes with continuous paths.

<u>Generalized Itô's Lemma:</u> Let

$$d\underline{X} = \underline{\underline{\sigma}} \cdot d\underline{W} + \underline{\mu}\,dt$$

Then: if $f(x,t): \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is a function

$$df(X_t, t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(t),t) \cdot dX_i(t) + \frac{\partial f}{\partial t} dt$$

$$+ \frac{1}{2} \sum_{ij=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(X(t),t)\,dt$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(t),t)\left(\sum_{k=1}^{m} \sigma_{ik} \cdot dW_k\right) +$$

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(X(t),t)\,\mu_i(t)\,dt + \frac{\partial f}{\partial t} \cdot dt$$

$$+ \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(X(t),t) \cdot dt$$

where $\boxed{a_{ij} = \sum_{k=1}^{m} \sigma_{ik} \sigma_{jk}}$

<u>Summarizing:</u>

If $d\underline{X} = \underline{\underline{\sigma}} \cdot d\underline{W} + \underline{\mu}\, dt$

$\underline{W} = (W_1 \cdots W_n)$ are iid BM.

$$\boxed{df(X,t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \left(\sum_{k=1}^{m} \sigma_{ik} \cdot dw_k \right) + \left(\frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right) dt}$$

"Generalised Itô's rule"

$$dX_i = \sigma_{ik} dW_k + \mu_i dt$$

$$\boxed{dX_i dX_j = a_{ij} dt}$$

<u>Diffusion Processes</u>. A diffusion process is an Itô process such that

$$\begin{cases} \sigma_{ik}(t) = \tilde{\sigma}_{ik}(X(t), t) \\ \mu_i(t) = \breve{\mu}_i(X(t), t) \end{cases}$$

In other words, Diffusion processes are Itô processes for which the local characteristics depend on the process (and not on the <u>past</u> for example).

Example:

$$dX(t) = \sigma X(t) \cdot dW_t + \mu X(t)\, dt$$

The local characteristics are $(\sigma X(t), \mu X(t))$, which are functions of position:

Note:

$$X(t) = X(0)\, e^{\sigma W_t - \frac{1}{2}\sigma^2 t + \mu t}$$

Multivariate stock model:

$$\frac{dX_i(t)}{X_i(t)} = \sum_{k=1}^{m} \sigma_{ik} \cdot dW_k + \mu_i\, dt$$

$$\hat{\sigma}_{ik} = \sigma_{ik} X_i \qquad \hat{\mu}_i = \mu_i X_i$$

The dynamics depend only on the current position. Thus $(X_1 \cdots X_n)$ is a multidimensional