Stochastic Calculus: Lecture 5

1. Itô's Stochastic Integral.

Def'n: $f(t)$: random function (process); $X(t)$ Brownian motion, defined on same pr. space. $\{f(t)\}$ is non-anticipative with respect to $\{X(t)\}$ if

$$f(t)\,; \quad X(t+s) - X(t)$$

are independent for all $t>0$ $s>0$.

We will define, for a given T (first)

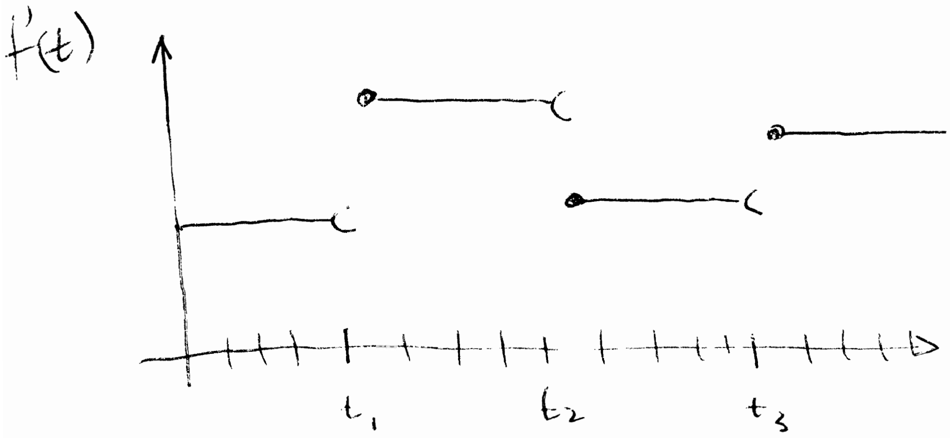
$$\int_0^T f(s)\,dX(s) = \lim_{\substack{\Delta t \to 0 \\ N \to \infty \\ \Delta T \cdot N = T}} \sum_{i=0}^{N-1} f(i\Delta T)\big(X(t+\Delta t) - X(t)\big)$$

$$= \lim_{\substack{\Delta T \to 0 \\ N \to \infty \\ \Delta T \cdot N = T}} \sum_{i=1}^{N-1} f_i \, \overrightarrow{\Delta X_i},$$

A. $f(t)$ is N.A and piecewise constant

In this case

$$f(t) = \begin{cases} f_i & t \in (i\Delta T, (i+1)\Delta T) \\ f_i \text{ indep if } X(s) - X(i\Delta T); & s > i\Delta T. \end{cases}$$



In this case, the limit as $\Delta T \to 0$ always exists since

$$\sum_{j=0}^{n} f(t_i + j\Delta t)\Big(X\big(t_i + \Delta t(j+1)\big) - X(t_i + \Delta t\, j)\Big)$$

$$= f(t_i)\big(X(t_{i+1}) - X(t_i)\big).$$

Therefore, successive refinement "inside" the partition defined by $f(\cdot)$, give rise to the same value

Furthermore:

$$E\left(\int_0^T f\,dX\right) = E\left(\sum_i f_i \,\Delta X_i\right)$$

$$= \sum E f_i \,\Delta X_i$$

$$= \sum E f_i \; E\Delta X_i$$

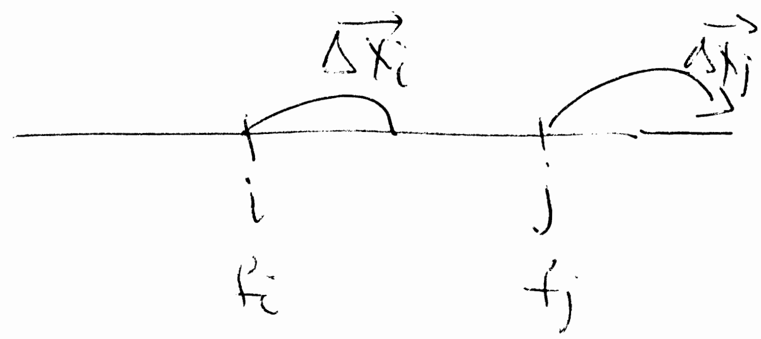
$$= \sum E f_i \cdot 0 = 0.$$

$$E\left(\int_0^T f\,dx\right)^2 = E\sum_i \sum_j f_i f_j \overrightarrow{\Delta X_i}\,\overrightarrow{\Delta X_j}$$

$$= \sum_i \sum_j E\, f_i f_j \overrightarrow{\Delta X_i}\,\overrightarrow{\Delta X_j}$$

$$= \sum_i E\, f_i^2 \Delta X_i^2 + 2\sum_{i<j} E f_i f_j \overrightarrow{\Delta X_i}\,\overrightarrow{\Delta X_j}$$

if $i<j$



ΔX_j indept of $f_i f_j \Delta X_i$

$$E(f_i f_j \Delta X_i \Delta X_j) = E(f_i f_j \Delta X_i)\, E\Delta X_j$$
$$= E(f_i f_j \Delta X_i)\cdot 0 = 0$$

$$E\left(\int_0^T f\,dx\right)^2 = \sum_i E\, f_i^2\, E\,\Delta X_i^2$$
$$= \sum_i E(f_i^2)\,\Delta T$$

$$\boxed{\mathbb{E}\left(\int_0^T f\,dX\right)^2 = \int_0^T \mathbb{E}\, f^2(s)\,ds}$$

This equation is known as the Itô Isometry. It relates the variance of the Itô integral to the expectation of $\int_0^T f^2(s)\,ds$.

B. The Itô integral ~~as~~ a process.

For piecewise constant $\{f(t)\}$, we can also view $\int_0^t f\,dX$ as a process

$$t \longmapsto \int_0^t f(s)\,dX(s)$$

This is a ~~continuous~~ process with continuous paths, since

$$\int_0^t f(s)\,dX(s) = \int_0^{t_i} f(s)\,dX(s) + f(t_i)\big(X(t) - X(t_i)\big)$$

$$t_i < t < t_{i+1}$$

and $X(t)$ is continuous.

Note: $\int_0^t f\,dX$ is a martingale

$$E\left[\int_0^t f\,dX \,\Big|\, \{f_u \; u \le s\}; \{X_u, u \le s\}\right]$$

$$= E\left[\int_0^s + \int_s^t \,\Big|\, \right]$$

$$= \int_0^s f \cdot dX + E\left[\int_s^t f\,dX \,\Big|\, f_u, X_u \; u \le s\right]$$

$$= \int_0^s f\,dX + \sum_i E\left[f_i \Delta\vec{X}_i \,\Big|\, f_u, X_u \, u \le s\right]$$

$$= \int_0^s f \cdot dX + \sum_i E\left[E\left[f_i \underline{\Delta X_i} \,\Big|\, f_u, X_u; u \le t_i\right] \Big| \cdot \right]$$

$$= \int_0^s f\,dX + 0$$ (ΔX_i is indep't of f_u $u \le t_i$)

$$= \int_0^s f\,dX.$$

Basically, the Itô integral is a rw with a "stochastic variance" determined by the integrand $f(t)$.

C. From piecewise constant to general N.A. integrands.

(C.1) **Doob's Inequality**. Let $Y_1 \cdots Y_N$ be a martingale sequence.

$$\boxed{P\left[\max_{n \le N} |Y_n| > \varepsilon\right] \le \frac{1}{\varepsilon^2} \mathbb{E}\left[Y_N^2\right]}$$

Proof:

$$\left\{\max_{n \le N} |Y_n| > \varepsilon\right\} = \bigcup_{n=1}^{N} \left\{|Y_n| > \varepsilon;\ |Y_1| \le \varepsilon, \ldots |Y_{n-1}| \le \varepsilon\right\}$$

$$P\left\{\max_{n \le N} |Y_n| > \varepsilon\right\} \le \sum_{n=1}^{N} P\left\{|Y_n| > \varepsilon;\ |Y_j| \le \varepsilon;\ j \le n-1\right\}$$

$$\le \sum_{n=1}^{N} \frac{1}{\varepsilon^2} E\left\{|Y_n|^2;\ |Y_j| \le \varepsilon,\ Y_j \le n-1;\ |Y_n| > \varepsilon\right\}$$

$$\le \frac{1}{\varepsilon^2} \sum_{n=1}^{N} E\left[|Y_N|^2; |Y_n| > \varepsilon;\ |Y_j| \le \varepsilon;\ j < n\right]$$

$$= \frac{1}{\varepsilon^2} E\left[|Y_N|^2 ; \max_{i \le N} |Y_n| > \varepsilon\right].$$

(Here, we used that

$$E[Y_N^2 \mid y_1 \dots y_n] \geq Y_n^2$$

(Jensen's Inequality).)

Doob's Inequality follows..

(C.2) Let $\{f_n(t)\}$ be a sequence of piecewise constant functions such ~~that~~

$$\lim_{n \to \infty} \int_0^T E\,|f_n(t) - f(t)|^2 = 0$$

(Thus $f_n \to f$ in the r.m.s. sense).

Generalizing Doobs inequality to continuous martingales: set

$$Y_t = \int_0^t f_n(s)\,dX(s) - \int_0^t f_m(s)\,dX_s$$

$$Y(t) = \int_0^t \left(f_n(s) - f_m(s)\right) \cdot dX(s)$$

$$= I_{f_n}(t) - I_{f_m}(t).$$

$\therefore$ From Doob, it follows that $\forall \varepsilon$

$$P\left[\max_{0 \le t \le T} \left|I_{f_n}(t) - I_{f_m}(t)\right| > \varepsilon\right] \le \frac{1}{\varepsilon^2} \int_0^T E\left|f_n(t) - f_m(t)\right|^2 dt.$$

The maximum deviation between the approximating stochastic integrals is controlled by the RMS distance between integrands.

Let $n_1, n_2, \ldots, n_k, \ldots$

be a sequence such that

$$\int_0^T E\left|f_{n_k}(s) - f_{n_{k+1}}(s)\right|^2 ds < \frac{1}{2^k}$$

It follows that

$$P\left[\sup_{t\le T}\left|I_{f_{n_k}}(t)-I_{f_{n_{k+1}}}(t)\right|>\frac{1}{2^{k/3}}\right]\le$$

$$\frac{2^{\frac{2}{3}k}}{2^k}=\frac{1}{2^{k/3}}.$$

In particular,

$$\sum P\left[\sup_{t\le T}\left|I_{f_{n_k}}(t)-I_{f_{n_{k+1}}}(t)\right|>\frac{1}{2^{k/3}}\right]<\infty$$

By the Borel-Cantelli Lemma

$$\text{Prob.}\left\{\sup_{t\ge T}\left|I_{f_{n_k}}(t)-I_{f_{n_{k+1}}}(t)\right|>\frac{1}{2^{k/3}},\ \text{for infinitely many } k\right\}=0$$

$\therefore$

$$\sum_k\left|I_{f_{n_{k+1}}}(t)-I_{f_k}(t)\right|<\infty$$

$\therefore$ $I_{f_k}(t)$ converges to a continuous function a.s.

We will not show (but this is standard) that this limit is independent of the subsequence, so $\int_0^t f(s)\,dX(s)$ is uniquely defined for functions that can be approximated in the RMS sense by piecewise constant functions.

$$t \longrightarrow I_f(t) = \int_0^t f \cdot dX$$

is the Ito integral of f with respect to X.

Properties

1. $I_f(t)$ is a martingale

$$\boxed{E\left[\int_0^t f(s)\,dX(s) \,\middle|\, f_w X_u ;\ u \le s\right] = \int_0^s f\,dX \quad \text{a.s.}}$$

(ii) $E \int f \, dX = 0$

(iii) $E \left(\int f \, dX \right)^2 = E \left(\int f^2 \right)$

(iv) $\left\{ t \to \int_0^t f \, dX \right\}$ has continuous paths.

Example: $\int_0^t X(t) \, dX(t) = I_t$. (Let's calc. this!)

$$I_t \sim \sum_i X_i (X_{i+1} - X_i)$$

$$= \sum_i X_i (X_{i+1} - X_i) - \sum X_{i+1}(X_{i+1} - X_i) + \sum X_{i+1} (X_{i+1} - X_i)$$

$$= - \sum_i (\Delta X_i)^2 + \sum_i X_{i+1} (X_{i+1} - X_i)$$

$$= - \sum_i (\Delta X_i)^2 + \sum_i X_{i+1}^2 - \sum_i X_{i+1} X_i$$

$$= - \sum_i (\Delta X_i)^2 + \sum_i (X_{i+1}^2 - X_i^2) - \sum (X_{i+1} - X_i) X_i$$

$$2 I_t \simeq -\sum (\Delta X_i)^2 + \sum_i X_{i+1}^2 - \sum X_i^2$$ (12)

$$\therefore \quad 2\int_0^t X_s\,dX_s = -t + X_t^2$$

$$\boxed{\int_0^t X_s\,dX_s = \frac{1}{2}\left(X_t^2 - t\right)}$$

Note: $X^2(t) - t$ is a Martingale.

In fact:

$$X^2(t) - t = \left[\left(\frac{d}{dx}\right)^2 \underbrace{e^{\lambda X(t) - \frac{1}{2}\lambda^2(t)}}_{(\text{Martingale})}\right]_{\lambda=0}.$$

2. Itô's Lemma

Let $f(x,t)$ be a "nice function"

$$\boxed{f(X_t,t) = f(X_s,s) + \int_s^t \left(\frac{\partial f}{\partial u}(X_u,u) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(X_u,u)\right)du + \int_s^t \frac{\partial f}{\partial x}(X_u,u)\cdot dX_u}$$

(13)

or,

$$\boxed{df(X_t,t) = \frac{\partial f}{\partial x}\cdot dX_t + \frac{\partial f}{\partial t}dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\cdot dt}$$

Proof: By Taylor expansion,

$$\Delta f(x,t) = \frac{\partial f}{\partial x}\cdot \Delta x + \frac{\partial f}{\partial t}\cdot \Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + \frac{\partial^2 f}{\partial x \partial t}\cdot \Delta x \Delta t + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\Delta t^2 + \cdots$$

If $t_1 < t_2 < t_3 \cdots$ is a partition

$f_i = f(X(t_i), t_i) \qquad X_i = X(t_i)$

$$f_{i+1} - f_i = \frac{\partial f}{\partial x}(X(t_i), t_i)\,\overrightarrow{\Delta X_i} + \frac{\partial f}{\partial t}(X_i, t_i)\,\Delta t_i + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\overrightarrow{\Delta X_i})^2 + O(|\overrightarrow{\Delta X}|\,\Delta t)$$

$$f(X(t),t) - f(X(s),s) \simeq \sum \left(\frac{\partial f}{\partial x}\right)_i (\overrightarrow{\Delta X_i}) + \sum\left(\frac{\partial f}{\partial t}\right)_i \Delta t_i + \frac{1}{2}\sum\left(\frac{\partial^2 f}{\partial x^2}\right)_i (\overrightarrow{\Delta X_i})^2 + \sum O(\Delta t)^{3/2}$$

Main term of importance:

$$A = \frac{1}{2} \sum_i \frac{\partial^2 f}{\partial x^2}(X(t_i),(t_i)) \left(X(t_{i+1}) - X(t_i)\right)^2$$

$\therefore$

$$A = \frac{1}{2} \sum_i \frac{\partial^2 f}{\partial x^2}(X(t_i), t_i)^2 \, \Delta t_i \; +$$

$$\frac{1}{2} \sum_i \frac{\partial^2 f}{\partial x^2}(X(t_i), t_i)\left((\Delta \vec{X}_i)^2 - \Delta t\right)$$

$$\simeq \frac{1}{2} \int_s^t \frac{\partial^2 f}{\partial X^2}(X(s), s) \cdot ds \; +$$

$$\frac{1}{2} \sum_i \frac{\partial^2 f}{\partial x^2}(X(t_i), t_i)\left(\Delta X_i^2 - \Delta t\right)$$

$$A \simeq \frac{1}{2} \int_s^t \frac{\partial^2 f}{\partial x^2}(X(s), s)\, ds + B$$

Claim: B ~~is this~~ has vanishing variance as $\Delta t \to 0$.

$$V(B) = \frac{1}{2} \sum_i E\left(\frac{\partial^2 f}{\partial x^2}\right)_i \left(\Delta \vec{X}_i^2 - \Delta t\right)$$

$$= \frac{1}{2} \sum_i E\left(\frac{\partial^2 f}{\partial x_i^2}\right) E\left(\Delta X_i^2 - \Delta t\right)$$

$$= 0.$$

$$E(B^2) = \frac{1}{4} \sum_{ij} E\left(\frac{\partial^2 f}{\partial x^2}\right)_i \left(\frac{\partial^2 f}{\partial x^2}\right)_j (\Delta X_i^2 - \Delta t) \cdot (\Delta X_j - \Delta t)^2$$

Due to the independence of Brownian increments ~~there~~

$$E\, B^2 = \frac{1}{4} \sum_i E\left[\left(\frac{\partial^2 f}{\partial x_i^2}\right)_i^2 \left(\Delta X_i^2 - \Delta t\right)^2\right]$$

$$= \frac{1}{4} \sum_i E\left(\frac{\partial^2 f}{\partial x^2}\right)_i^2 E\left(\Delta X_i^2 - \Delta t\right)^2$$

$$= \frac{1}{4} \sum_i E\left(\frac{\partial^2 f}{\partial x^2}\right)_i^2 \Delta T^2 \cdot E\left(Y_i^2 - 1\right)^2$$

$$\left[E\,(Y^2-1)^2 = E\,Y^4 - 2\,E\,Y^2 + 1 = 3 - 2 + 1 = 2\right]$$

$$= \frac{1}{2} \sum_i E\left(\frac{\partial^2 f}{\partial x^2}\right)_i^2 \Delta T^2$$

$$E\,B^2 \cong \frac{\Delta T}{2}\,E\left(\int_0^T\left(\frac{\partial^2 f}{\partial x^2}\right)^2 dy\right) = O(\Delta T)$$

$\therefore$ It follows that

$$\boxed{\begin{aligned} &\cancel{f(t)} \\ &f(X(t),t) - f(X(s),s) = \int_s^t \frac{\partial f}{\partial x}(X(u),u)\,dX(u) + \\ &+ \int_s^t \frac{\partial f}{\partial t}(X(u),u)\cdot du \; + \\ &+ \frac{1}{2}\int_s^t \frac{\partial^2 f}{\partial x^2}(X(u),u)\,du \end{aligned}}$$

In shorthand:

$$df = \frac{\partial f}{\partial x}\cdot dx + \frac{\partial f}{\partial t}\,dt + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\,dt$$

Example: $\int X_s^2 \, dX(s) = ?$

$$d X^3(t) = 3 X^2(t) \cdot dX(t) + \frac{3}{2} X(t)\, dt$$

$$X^3(t) = 3 \int_0^t X^2(s)\, dX(s) + \frac{3}{2} \int_0^t X(s)\, ds$$

$$\boxed{\int_0^t X^2(s)\, dX(s) = \frac{1}{3} X^3(t) - \frac{1}{2} \int_0^t X(s)\, ds}$$

Example. $f(x,t) = e^{\lambda x - \frac{1}{2}\lambda^2 t}$

$$d f(X_t, t) = \lambda f \, dX - \frac{1}{2} \lambda^2 f \, dt + \frac{1}{2} \lambda^2 f \, dt$$

$$\boxed{df = \lambda f \, dX}$$