Lecture 4: Stochastic Calculus

1. Brownian Quadratic Variation

Let $\{X(t): 0<t<T\}$ be a Brownian motion. The quadratic variation of $X(t)$ is

$$Q_T(X) = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \left(X(i\Delta t) - X((i-1)\Delta t)\right)^2$$

$$(n\Delta T = T)$$

It is a rigorous mathematical result that

$$\boxed{Q_T(X) = T \quad \text{with prob. 1 over all paths.}}$$

This means that $Q_T(X)$ exists for any BM path (almost any, anyway) and is equal to T. Before examining this, let us observe that is $f(t)$ is a generic differentiable function

~~(Δf)~~

$$f(i\Delta T) - f((i-1)\Delta T) = \int_{(i-1)\Delta T}^{i\Delta T} f'(s)\,ds$$

$$\therefore \quad (\Delta f)^2 = \left| \int_{(i-1)T}^{i\Delta T} f'(s)\,ds \right|^2$$

$$\leq (\Delta T)^2 \left(\max_{s \leq T} |f'(s)|^2\right).$$

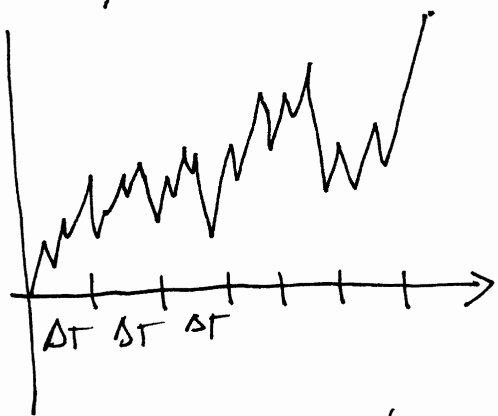
$$\therefore \quad \sum_{i=1}^{n} (\Delta f_i)^2 \leq (\Delta T)^2 \cdot \left(\max_{s \leq T} |f'(s)|^2\right)$$

If the first derivative is bounded, then

$$\lim_{\Delta T \to 0} \sum |\Delta f_i|^2 = 0.$$

Thus, the finiteness of the QV reflects the non-differentiability of BM. The other interesting idea comes from <u>statistics</u>.

If you have a BM path we



can estimate the variance of incre ments by the

~~The va~~ estimator

$$\hat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\Delta X_i - \overline{\Delta X_i} \right)^2 \qquad \Delta X_i = \left(X_i(i\Delta T) - X((i-1)\Delta T) \right).$$

Recall that $\Delta X_i = \nu_i \sqrt{\Delta T}$ $\left(\begin{matrix} \nu_i \in N(0,1) \\ \text{i.d.} \end{matrix} \right)$

$$\overline{\Delta X_i} = \frac{1}{n} \sum \Delta X_i = \frac{1}{n} \sum \nu_i \sqrt{\Delta T}$$

$$\hat{\sigma}_n^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left(\nu_i \sqrt{\Delta T} - \overline{\nu} \sqrt{\Delta T} \right)^2$$

$$= \frac{\Delta T}{n-1} \sum_{i=1}^{n} \left(\nu_i - \overline{\nu} \right)^2$$

Since ν_i are iid $\hat{\sigma}_n^2 \longrightarrow \Delta T$ as $n \to \infty$.

this is the limit in which $n \to \infty$ and we estimate the variance of an increment. Instead, here we do a refinement, keeping time fixed and increasing the number of intervals. Consider <u>dyadic partitions</u>

$$t_{n,j} = \frac{j}{2^n}\cdot T \qquad 0 < j \le 2^n.$$

$$\sum_{j=1}^{2^n} (\Delta X_{j,n})^2 = \sum_{j=1}^{2^n} \nu_{n,j}^2 \left(\sqrt{\frac{T}{2^n}}\right)^2$$

$$= T\left(\frac{1}{2^n}\sum_{j=1}^{2^n} \nu_{n,j}^2\right).$$

Almost sure convergence follows from the Law of Large numbers for triangular arrays.

<u>LLN "Triangular arrays".</u>

<u>Theorem</u>: Let $\{X_{N,j} \ (1 \le j \le N)\}$ be IID

$E X_{N,1} = 0 \qquad E X_{N,1}^4 < \infty$ then

$$\boxed{\frac{1}{N}\sum_{1}^{N} X_{N,j} \xrightarrow{a.s.} 0}$$

a.s. = almost surely

$$P\left\{\left|\frac{1}{N}\sum_{1}^{N} X_{Nj}\right| > \varepsilon\right\} \leq$$

$$\frac{1}{\varepsilon^4}\,\mathbb{E}\left\{\frac{1}{N^4}\left(\sum X_{Nj}\right)^4\right\}$$

$$= \frac{1}{\varepsilon^4 N^4}\left(\sum_{1}^{N} E X_{Nj}^4 + \sum_{jj'}^{N} E\left(X_{Nj}^2 X_{Nj'}^2\right)\right)$$

NO ODD TERMS, SINCE $E X_{Nj} = 0$

$$= \frac{1}{\varepsilon^4 N^4}\left(N\,E X_{N1}^4 + N^2\left(E X_{N1}^2\right)^2\right)$$

$$\leq \frac{C}{\varepsilon^4 N^2}$$

$\therefore$ Take $\varepsilon = \frac{1}{N^\gamma}$

$$\therefore\ P\left\{\left|\frac{1}{N}\sum_{1}^{N} X_{N,j}\right| > \varepsilon\right\} \leq \frac{C}{N^{2-4\gamma}}$$

$$2 - 4\gamma > 1 \implies 1 > 4\gamma \qquad \boxed{\gamma < \frac{1}{4}}$$

Choose $\gamma = 1/8$

$$\boxed{P\left\{\left|\frac{1}{N}\sum_{1}^{N} X_{Nj}\right| > \frac{1}{N^{1/8}}\right\} \leq \frac{C}{N^{3/2}}}$$

Since the right-hand side defines a convergent series, by the Borel - Cantelli Lemma,

$$P\left\{ \left| \frac{1}{N} \sum_{1}^{N} X_{N,j} \right| > \frac{1}{N^{1/8}} \text{, infinitely often} \right\} = 0.$$

$\therefore$ It follows that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{1}^{N} X_{N,j} = 0 .$$

Going back to QV., this shows that

$$\lim_{n \to \infty} \sum_{1}^{2^n} \left(X\left(\frac{jT}{2^n}\right) - X\left(\frac{(j-1)T}{2^n}\right) \right)^2$$

$$= T \qquad a.s.$$

If we take non-dyadic refinements the trick is to compare to dyadics.

This further refinement is left as an exercise to the interested student.

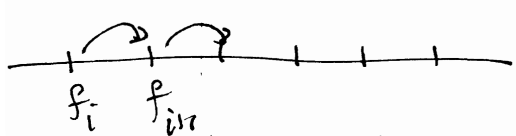
2. Stochastic Integrals

Let $f(t)$ be a function of t, (deterministic!) and let $\{X(t)\}$ be a Brownian path. Choose a partition of $(0,T)$ and set

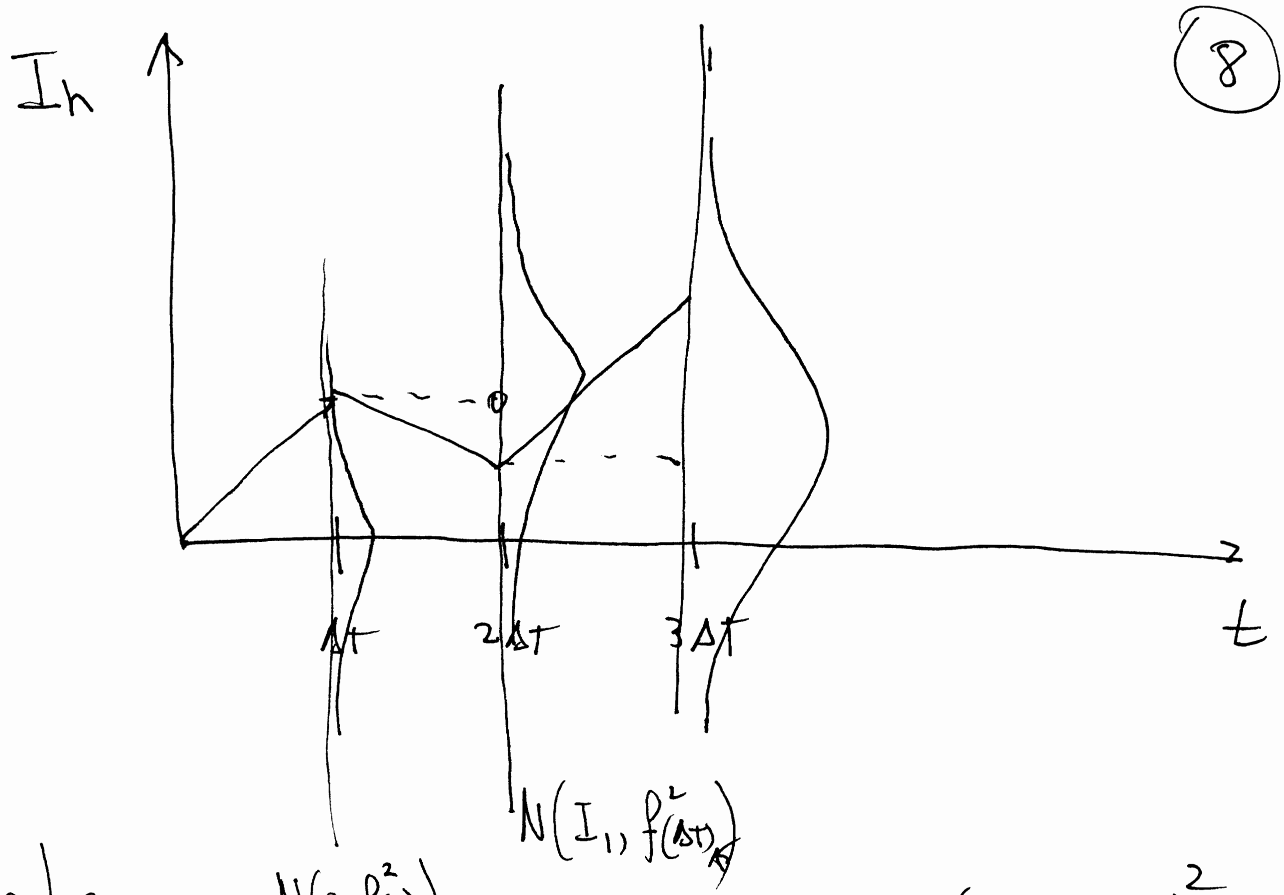
$$I_n = \sum_{i=1}^{n} f\big((\Delta T)(i-1)\big) \cdot \big(X(i\Delta T) - X((i-1)\Delta T)\big)$$

$(n \le N)$

$(N\Delta T = T)$ In this sum, the evaluation of $f(t)$ is done before the jump.



Let us fix ΔT and move n first, $1 \le n \le N$



We have:

$$P[\,I_{n+1}=x \mid I_n\,] = \frac{e^{-\frac{(x-I_n)^2}{2f^2(n\Delta T)\Delta T}}}{\sqrt{2\pi f^2(n\Delta T)\Delta T}}$$

(In the sense of pdf's.) Thus, if we view these r.v.s as a discrete process, they look like a BM with a time-dependent variance, or a random walk with a time-dependent variance.

First moment = 0.

$$E\, I_n = E\left[\sum_{j=0}^{n-1} f((j-1)\Delta T)\cdot \Delta \vec{X}_j\right]$$

$$= \sum_{j=0}^{n-1} E\left(f((j-1)\Delta T)\, \Delta\vec{X}_j\right)$$

$$= \sum_{j=0}^{n-1} \left(f((j-1)\Delta T)\right)\cdot E\, \Delta\vec{X}_j$$

$$= 0.$$

Variance

$$E\, I_n^2 = \sum_{jj'} E\, f_j f_{j'}\, \Delta\vec{X}_j\, \Delta\vec{X}_{j'}$$

$$= \sum_{jj'} f_j f_{j'}\, E\, \Delta\vec{X}_j\, \Delta\vec{X}_{j'}$$

$$= \sum_{j=0}^{n-1} f_j^2\, E\, \Delta X_j^2$$

$$= \sum_{j=0}^{n-1} f^2(j\Delta T)\, \Delta T.$$

$$\simeq \int_0^{n\Delta T} f^2(s)\, ds.$$

Third moment = 0.

$$E I_n^3 = E \sum_{abc} f_a f_b f_c \, \vec{\Delta X}_a \, \vec{\Delta X}_b \, \vec{\Delta X}_c$$

Since odd moments are zero $E I_n^3 = 0$.

Fourth moment:

$$E I_n^4 = \sum_a f_a^4 \, E(\vec{\Delta X}_a)^4 +$$

$$+ 6 \sum_{a \neq b} f_a^2 f_b^2 \, E(\vec{\Delta X}_a^2 \, \vec{\Delta X}_b^2)$$

$$\boxed{(x_1 + x_2 + \cdots + x_N)^p = \sum_{|\alpha| = p} \frac{p!}{\alpha_1! \cdots \alpha_N!} x_1^{\alpha_1} \cdots x_N^{\alpha_N}}$$

$$\frac{4!}{2!\,2!} = \frac{4 \times 3 \times 2 \times 1}{2 \times 2} = 6.$$ (This is where 6 comes from).

Also: $E \Delta X^4 = 3 \Delta T^2$

$$\therefore \quad E I_n^4 = 3 \left(\sum_a f_a^4 \Delta T^2 + 2 \sum_{a \neq b} f_a^2 f_b^2 \Delta T^2 \right)$$

$$= 3 \left(\int f^2(a \Delta T) \, \Delta T \right)^2$$

$$E I_n^4 \simeq 3 \left(\int_0^{n \Delta T} f^2(s) \, ds \right)^2.$$

This calculation suggests that

$$I_f(T) \equiv \lim_{\substack{n \to \infty \\ \Delta T \to 0 \\ n\Delta T = T}} \sum_{j=0}^{n-1} f(j\Delta T)\left[X((j+1)\Delta T) - X(j\Delta T)\right]$$

exists a.s. and is a Gaussian r.v. such that

$$\begin{cases} E\, I_f(T) = 0 \\ E\,(I_f(T))^2 = \int_0^T f(s)^2\, ds \end{cases}$$

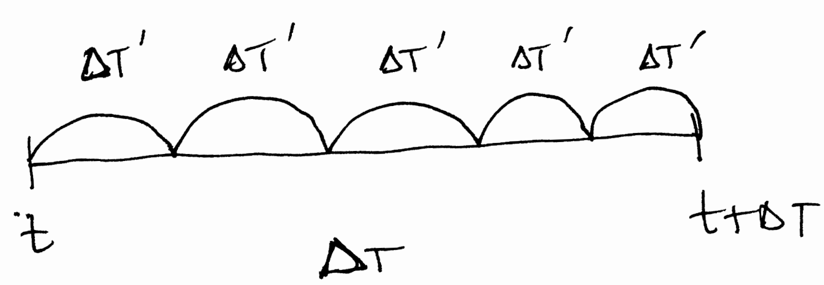
We call $I_f(T)$ the stochastic integral of f (with respect to BM).

~~$I_f(T)$ as a function of T. Since~~

The proof of existence of $I_f(T)$ is done by observing that if we take successive refinements of the partition, the variance of ~~Rg~~ the difference between two approximating funcs is small.

Assume ~~that~~ $f(t)$ is continuous.

$$\Delta T = \frac{1}{2^n} \qquad \Delta T' = \frac{1}{2^{n+m}}$$



$$\xi = f(t)\Big(X(t+\Delta t) - X(t)\Big) - \sum_i f(t_i)(\Delta X_i) =$$

$$= \sum_i \big(f(t_i) - f(t)\big)(\Delta X_i)$$

$$\mathrm{Var}(\xi) = \sum_i \big(f(t_i) - f(t)\big)^2 (\Delta T')$$

$$\leq \Delta T \Big(\max_{t<s<t+\Delta t} |f(s) - f(t)|^2 \Big)$$

The difference $\xi_{n,m}$ between two approximations func. with $\Delta T = \frac{1}{2^n}$ $\Delta T' = \frac{1}{2^{n+m}}$

is such that

$$\mathrm{Var}(\xi) \leqslant \sup_{\substack{|s-s'|< \frac{1}{2^{n+m}} \\ s \leq T \\ s' < T}} |f(s)-f(s')|^2$$

Therefore, the variance ~~of differences~~ tends to zero with successive refinements. Using Borel-Cantelli we can show ~~that~~ there is convergence to a r.v. $I_f(T)$ with probability 1.

$$I_f(T) \sim N\left(0, \int_0^T f^2(s)\,ds\right).$$

$I_f(t)$ as a process in t.

We can define simultaneously $I_f(t)$ if $t \in \{ j/2^n \, T ; \, j \in \mathbb{N}, n \in \mathbb{N} \}$, since the countable union of zero-probability events has zero probability.

Furthermore, ~~according to the 4th moment calculation~~

For all $t \in \{ j/2^n \, T \}$

(i) $I_f(t) \sim N\left(0, \int_0^t f^2 \, ds\right)$

(ii) $I_f(t)$, $I_f(t+\Delta t) - I_f(t)$ are independent, gaussian.

In particular

$$E\left(I_f(t+\Delta t)-I_f(t)\right)^4 = 3\left(\int_t^{t+\Delta t} f^2\right)^2$$

$$\simeq O(\Delta t)^2$$

This means that there is a ~~continuous~~ unique extension of $I_f(t)$ to $t \in (0,T)$ as a continuous function (a.s.). We write:

$$I_f(t) \doteq \int_0^t f(s)\,dX(s)$$

~~$I_f(t)$ is sometimes called colored~~ $I_f(t)$ is essentially a Brownian motion in which the variance is time-dependent.

$I_f(t)$ is called a stochastic integral.

Let $Y(\theta)$ be a Brownian motion. Set

$$\xi_t = Y\left(\int_0^t f^2(s)\, ds \right)$$

then:

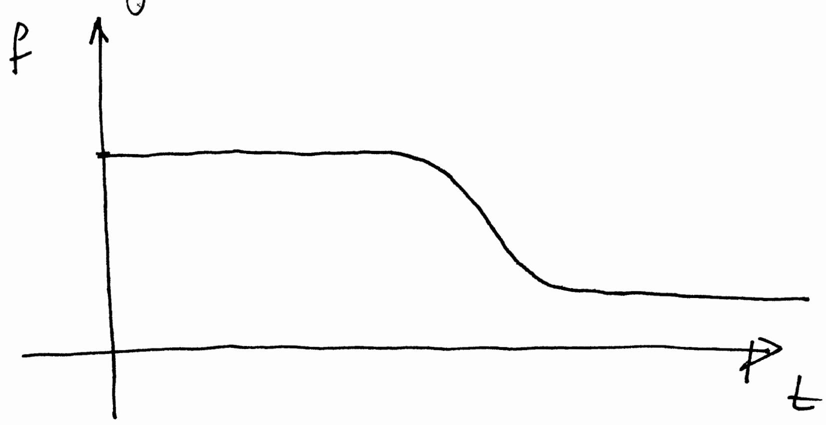
(i) ξ_t is Gaussian

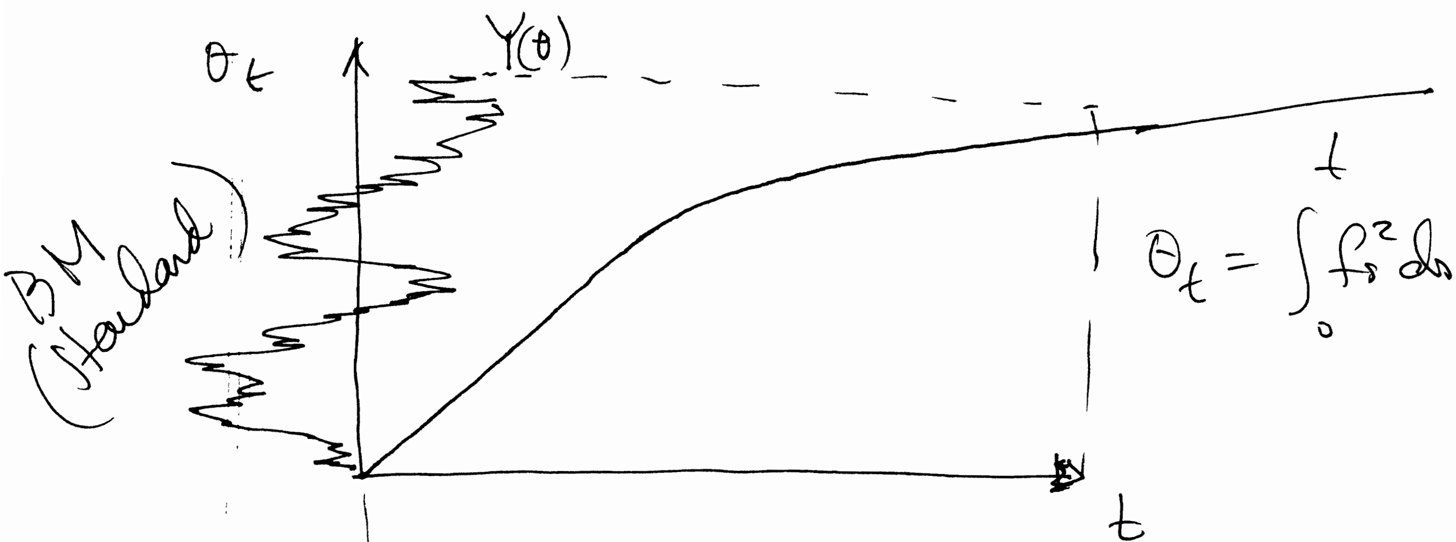
(ii) $\xi_{t+\Delta t} - \xi_t$ and ξ_t are independent.

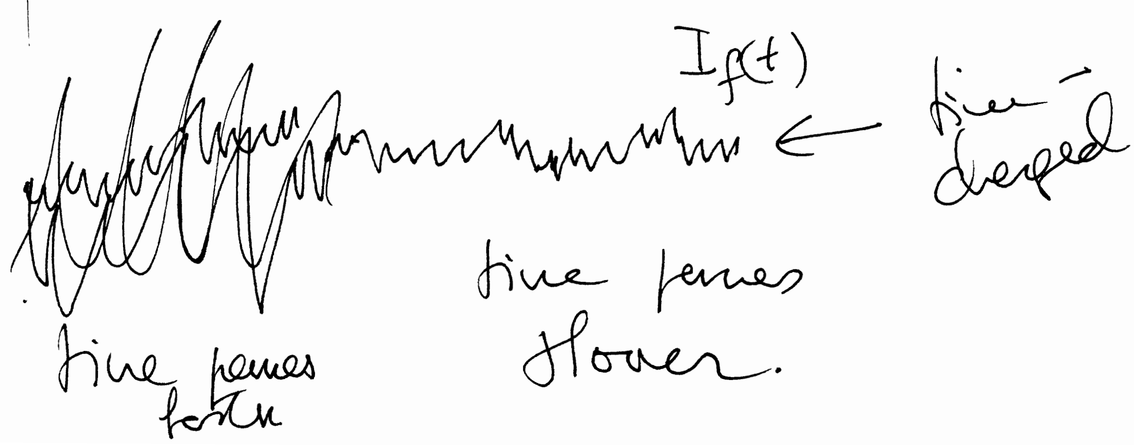
(iii) $E\, \xi_t^2 = \int_0^t f(s)^2\, ds$.

Thus, the stochastic integral

with respect to a function $f(t)$, corresponds, as a process, to a Brownian motion with a time change.







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Stochastic Integral with respect to a deterministic function of time $f(t)$ is nothing but a time-changed BM.

<u>Application</u>: Suppose you have a model for the ~~daily~~ intraday trading volume of a stock

$$V_t = \frac{\text{volume @ time } t}{\text{Average daily Volume}}$$

Assume that the volatility of a stock price is constant or a ~~per share model~~ "per-share hesis".

Let

$$\theta_t = \int_0^t V_s \, ds$$

be the total (cumulative) volume traded until time t. If the statistics of price changes (returns) are the same on a (per-volume / per-share) basis, we have

$$S_\theta = S_0 e^{\sigma Y(\theta) - \frac{1}{2}\sigma^2 \theta}$$

as a possible simple model for price changes. In "actual time":

$$S_t = S_0 e^{\sigma Y\left(\int_0^t f(s)^2 ds\right) - \frac{1}{2}\sigma^2 \int_0^t f(s)^2 ds}$$

$$\boxed{S_t = S_0 e^{\sigma \int_0^t f(s) dX(s) - \frac{1}{2}\sigma^2 \int_0^t f(s)^2 ds} \qquad f(s) = \sqrt{V(s)}}$$

Volatility in the period $(t, t+\Delta t)$ is

$$\frac{1}{\Delta T}\,\sigma^2 \int_t^{t+\Delta T} V(s)\,ds \cong \sigma^2 V(t)$$

This model matches <u>qualitatively</u> market data.