Stochastic Calculus, Lecture 12

Fundamental Solutions, Fokker Planck Equations, Invariant Measure, Conditioning

Let (X_t) be a diffusion process

$$\boxed{dX_t = \sigma(X_t,t)\cdot dW_t + \mu(X_t,t)\,dt}$$

with associated infinitesimal generator

$$\mathcal{L}\phi = \frac{1}{2}\sigma^2(x,t)\,\phi_{xx} + \mu(x,t)\,\phi_x$$

Let $\quad \pi(x,t,y,T) = \mathrm{Prob}\{X_T = y \mid X_t = x\}$

Then

$$(1) \quad \begin{cases} \dfrac{\partial \pi}{\partial t} + \mathcal{L}\pi = 0 & 0<t<T \\ \pi\big|_{t=T} = \delta(y-x) \end{cases}$$

This is the backward Fokker Plank equation

This follows immediately from

$$E\left[F(X_T) \mid X_t = x\right] = \phi(x,t)$$

$$\frac{\partial \phi}{\partial t} + \mathcal{L}\phi = 0$$

If we represent $\phi(x,t)$ as

$$\phi(x,t) = \int F(y)\, \pi(x,t;y,T)\, dy$$

$$\int F(y) \left(\frac{\partial}{\partial t} + \mathcal{L}_x\right) \pi(x,t;y,T)\, dy = 0$$

Since this must hold for all $F(x)$, we conclude that (1) holds.

We now derive a PDE for

$\pi(x\,t;y\,T)$ in the "forward variables" $(y;T)$.

We start with the identity

$$\pi(x,t,y,T) = \int \pi(x t; z, \theta)\,\pi(z,\theta; y,T)\,dz$$

(Chapman-Kolmogorov Equation)

which is valid for all θ, $t<\theta<T$

Differentiating wrt θ

$$0 = \int \frac{\partial \pi}{\partial \theta}(x t\; z \theta)\,\pi(z\theta, y,T)\,dz + \int \pi(x t; z\theta)\,\frac{\partial}{\partial \theta}\pi(z,\theta; y,T)\,dz$$

$$0 = \int \frac{\partial}{\partial\theta}\pi(x\,t, \cancel{z}\,\theta)\ \pi(z\,\theta, y\,T) +$$

$$+ \int \pi(x,t; z\,\theta)\left(-\frac{1}{2}\sigma^2 \pi_{zz} - \mu\,\pi_z\right)$$

$$= \int \frac{\partial\pi}{\partial\theta}(x\,t, z; \theta)\,\pi(z\,\theta, y\,T) +$$

$$+ \int -\frac{1}{2}\sigma^2(z,\theta)\,\pi(x\,t; z,\theta)\ \pi_{zz}(\cancel{z\,\theta}, \cancel{y\,T}) -$$

$$-\mu(z,\theta)\,\pi(x\,t, z,\theta)\ \pi_z(z,\theta; y,T)$$

(integration by parts)

$$= \int \frac{\partial\pi}{\partial\theta}(x\,t, z,\theta)\,\pi(z,\theta; y,T) +$$

$$\int \left[-\frac{1}{2}\left(\sigma^2(z,\theta)\,\pi(x\,t, z\,\theta)\right)_{zz} + \left(\mu(z\,\theta)\,\pi(x\,t\,z,\theta)\right)_z \right.$$

$$\times\ \pi(z\,\theta, y\,T)$$

If we set

$$\mathcal{L}^*_{z,\theta} = \left(\frac{1}{2}\sigma^2(z,\theta)\,\phi(z,\theta)\right)_{zz} - (\mu\phi)_z$$

then

$$0 = \int \left(\frac{\partial\pi}{\partial\theta}\pi(x,t,z,\theta) - \mathcal{L}^*_{z,\theta}\pi(x,t,z,\theta)\right)\cdot \pi(z,\theta,y,T)\,dz$$

Now, let ~~T~~ $\theta \to T$

$$0 = \int \frac{\partial\pi}{\partial T}(x,t,z,T) - \mathcal{L}^*_{z,T}\pi(x,t,z,T)\cdot \delta(z-y)\,dz$$

$$\boxed{\frac{\partial\pi}{\partial T} = \mathcal{L}^*_{y,T}\pi}$$

(Forward Fokker Planck Equation)

The forward FPE describes the evolution of the PDF of the diffusion in the <u>forward variable</u>

<u>Example:</u> $X_t \sim$ O.U.

$$dX = -\kappa X\, dt + \sigma\, dW$$

$$\mathcal{L}\phi = \frac{1}{2}\sigma^2 \phi_{xx} - \kappa x\, \phi_x$$

$$\mathcal{L}^*\phi = \left(\frac{1}{2}\sigma^2\phi\right)_{xx} + \left(\kappa x\phi\right)_x$$

$$\boxed{\mathcal{L}^*\phi = \frac{1}{2}\sigma^2\phi_{xx} + \left(\kappa x\phi\right)_x}$$

$\pi(x,t;y,T)$ satisfies

$$\frac{\partial \pi}{\partial T} = \mathcal{L}^*\pi$$

In particular if $\cancel{T} \to \infty$,

let

$$\bar{\pi}(x\,t, y) = \lim_{\cancel{T} \to \infty} \pi(x, t; y, T)$$

$$\mathcal{L}^*_{yt}\ \bar{\pi} = 0$$

$$\frac{1}{2}\sigma^2 \bar{\pi}_{yy} + (K y \bar{\pi})_y = 0$$

$$\therefore \quad \frac{1}{2}\sigma^2 \bar{\pi}_y + K y \bar{\pi} = 0$$

$$\frac{\bar{\pi}_y}{\bar{\pi}} = -\frac{2Ky}{\sigma^2}$$

$$\ln \bar{\pi} = -\frac{Ky^2}{\sigma^2} + c$$

$$\bar{\pi} = \bar{C}\, e^{-\frac{Ky^2}{\sigma^2}}$$

$$\boxed{\begin{aligned} \bar{\pi}(y) &= \frac{1}{\sqrt{2\pi\sigma_e^2}}\, e^{-\frac{y^2}{2\sigma_e^2}} \\ \sigma_e^2 &= \frac{\sigma^2}{2\kappa} \end{aligned}}$$

We recover from the FFPE the long-time behavior of the process X. (Which we knew by explicit solution of SDE!)

Ex 2: CIR Process

$$dX = \alpha\sqrt{X}\, dW + \kappa(\theta - X)\, dt$$

$$\mathcal{L}\phi = \frac{1}{2}\alpha^2 X \phi_{XX} + \kappa(\theta - X)\phi_X$$

$$\mathcal{L}^*\phi = \left(\frac{1}{2}\alpha^2 X\phi\right)_{XX} - \left(\left(\kappa(\theta - X)\right)\phi\right)_X$$

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The equilibrium distribution associated with the CIR process r $\bar{\pi}(y)$, where

$$\mathcal{L}^{Y*}\bar{\pi}(y) = 0$$

$$\left(\frac{1}{2}\alpha^2 y\bar{\pi}\right)_{yy} - \left(\left(\kappa(\theta - y)\right)\bar{\pi}\right)_y = 0$$

$$\left(\frac{1}{2}\alpha^2 y\bar{\pi}\right)_y - \kappa(\theta - y)\bar{\pi} = 0$$

$$(y\bar{\pi})_y = \frac{2\kappa}{\alpha^2}(\theta - y)\bar{\pi}$$

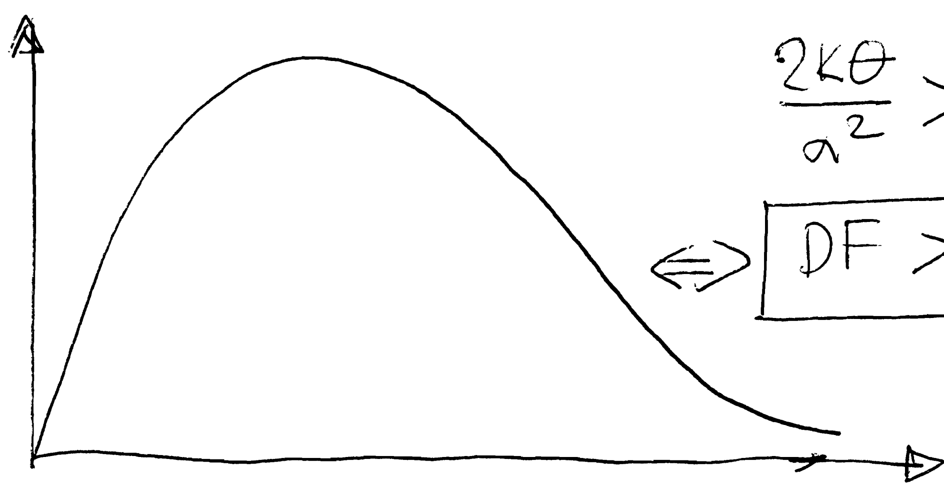
$$= \frac{2\kappa\theta}{\alpha^2 y}(y\bar{\pi}) - \frac{2\kappa}{\alpha^2}y\bar{\pi}$$

$$\frac{(y\bar{\pi})_y}{y\bar{\pi}} = \frac{2k\theta}{\alpha^2 y} - \frac{2k}{\alpha^2}$$

$$\ln y\bar{\pi} = \frac{2k\theta}{\alpha^2}\ln y - \frac{2k}{\alpha^2}y + C$$

$$\therefore \quad y\bar{\pi} = C\cdot y^{\frac{2k\theta}{\alpha^2}} e^{-\frac{2k}{\alpha^2}y}$$

$$\boxed{\bar{\pi}(y) = C\, y^{\frac{2k\theta}{\alpha^2}-1} e^{-\frac{2k}{\alpha^2}y}}$$



$$\frac{2k\theta}{\alpha^2} > 1$$

$$\Leftrightarrow \boxed{DF > 2}$$

This is a so-called Gamma distribution.

The constant of integration can be computed easily from

$$\Gamma(p) = \int_0^\infty x^p e^{-x}\,dx$$

$$\Gamma(0) = \int_0^\infty e^{-x}\,dx = 1$$

$$\Gamma(1) = \int_0^\infty x e^{-x}\,dx = -xe^{-x}\Big]_0^\infty + \int_0^\infty e^{-x}\,dx$$

$$= 1$$

$$\Gamma(n) = n!$$

$$\int_0^\infty y^{\frac{2K\theta}{\alpha^2}-1} e^{-\frac{2K}{\alpha^2}y}\,dy =$$

$$\left(\frac{\alpha^2}{2k}\right)^{\frac{2K\theta}{\alpha^2}} \int_0^\infty z^{\frac{2K\theta}{\alpha^2}-1} e^{-z}\,dz$$

$$= \left(\frac{\alpha^2}{2K}\right)^{\frac{2K\theta}{\alpha^2}} \Gamma\left(\frac{2K\theta}{\alpha^2} - 1\right)$$

$$\boxed{\bar{\pi}(y) = \frac{1}{\left(\frac{\alpha^2}{2K}\right)^{\frac{2K\theta}{\alpha^2}} \Gamma\left(\frac{2K\theta}{\alpha^2} - 1\right)} \, y^{\frac{2K\theta}{\alpha^2} - 1} e^{-\frac{\alpha^2}{2K} y}}$$

Notice that if $\frac{2K\theta}{\alpha^2} \leq 1$, the boundary conditions which we need to ~~enfor~~ derive the FP equation are not satisfied.

This means that we need to impose furthe conditions on the SDE near $x=0$.

The Brownian Bridge

$B_{t,T}$ = Brownian motion conditional on $W_T = 0$.

$$P_0[W_t \in A \mid W_T = 0] =$$

$$= \frac{P[W_t \in A ; W_T = 0]}{P[W_T = 0]}$$

$$= \frac{\displaystyle\int_A \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \frac{e^{-\frac{x^2}{2(t-T)}}}{\sqrt{2\pi(T-t)}} \, dx}{\dfrac{1}{\sqrt{2\pi T}}}$$

$$= \int_A e^{-\frac{1}{2}x^2\left(\frac{1}{t}+\frac{1}{T-t}\right)} \frac{dx}{\sqrt{2\pi \frac{(T-t)t}{\cancel{(T-t)}}}}$$

$$= \int_A e^{-\frac{1}{2}x^2 \frac{T}{t(T-t)}} \frac{dx}{\sqrt{2\pi \frac{(T-t)t}{T}}}$$

$$\sigma^2(B_{t,T}) = \frac{t(T-t)}{T} = \frac{t}{T}\left(1-\frac{t}{T}\right)$$

<u>What about transitions?</u>

$$P_{xs}\left[W_t \in A \mid W_T = 0\right] =$$

$$\frac{P_{xs}\left[W_t \in A;\ W_T = 0\right]}{P_{xs}\left[W_T = 0\right]}$$

$$= \frac{\displaystyle\int_A e^{-\frac{(y-x)^2}{2(t-s)}}\, e^{-\frac{y^2}{2(T-t)}} \frac{dy}{2\pi\sqrt{(t-s)(T-t)}}}{\dfrac{e^{-\frac{x^2}{2(T-s)}}}{\sqrt{2\pi(T-s)}}}$$

$$\pi_T(x,s,;y,t) = \frac{1}{\sqrt{2\pi\left(\frac{(t-s)(T-t)}{(T-s)}\right)}}\, e^{-\frac{(y-x)^2}{2(t-s)} - \frac{y^2}{2(T-t)} + \frac{x^2}{2(t-s)}}$$

$$\cancel{\boxed{\Pi_T(x\,s;y,t) = \frac{\pi(x\,s,\,y,t)\,\pi(y\,t,0,T)}{\pi(x\,s,0\,T)}}}$$

$$P[B_{t,T} \in A] = \frac{P[W_t \in A ; W_T = 0]}{P[W_T = 0]}$$

$$= \lim_{\varepsilon \downarrow 0} \frac{E\left[W_t \in A ; e^{-\frac{1}{2\varepsilon} W_T^2}\right]}{E\left[\cancel{W_t}\, e^{-\frac{1}{2\varepsilon} W_T^2}\right]}$$

$$V(x,t) = \frac{1}{2\varepsilon} \delta(t-T) x^2$$

$$\zeta_\varepsilon(x,t) = E_{xt}\left[e^{-\frac{1}{2\varepsilon} W_T^2}\right]$$

$$= \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\varepsilon} - \frac{(y-x)^2}{2(T-t)}} \frac{dy}{\sqrt{2\pi(T-t)}}$$

$$\frac{\partial \zeta_\varepsilon(x,t)}{\partial x}\frac{1}{\zeta_\varepsilon(x,t)} = \frac{\frac{1}{\sqrt{2\pi\epsilon}}\int\left[\frac{y-x}{2(T-t)}\right]e^{-\frac{y^2}{2\epsilon}}e^{-\frac{(x-y)^2}{2(T-t)}}dy}{\frac{1}{\sqrt{2\pi\epsilon}}\int e^{-\frac{y^2}{2\epsilon}-\frac{(x-y)^2}{2(T-t)}}dy}$$

$$\xrightarrow[\varepsilon\downarrow 0]{} \frac{\int \frac{y-x}{2(T-t)}\,\delta(y)\,e^{-\frac{(x-y)^2}{2(T-t)}}}{\int \delta(y)\,e^{-\frac{(x-y)^2}{2(t-t)}}dy}$$

$$= \frac{-\frac{x}{2(t-t)}\,e^{-\frac{x^2}{2(T-t)}}}{e^{-\frac{x^2}{2(t-t)}}} = \frac{-x}{2(T-t)}$$

This implies that the Brownian Bridge satisfies the diffusion equation

$$\boxed{dX_t = \frac{-X_t}{(T-t)}\,dt + dW_t}$$

Thus, the BB is an A-R(1) process with a speed of mean-reversion that increases as $t \to T$.

The FFB equatn is

$$\frac{\partial \pi(x,t,y,t')}{\partial t'} = \frac{1}{2}\,\pi(x,t,y,t')_{yy} + \left(\frac{y}{(T-t')}\pi(x,t,y,t')\right)_y$$

Conditioning a BM to stay positive for all times

$$P\left[X_t \in A\right] = \frac{P\left[\cancel{W}_t \in A ; \min\limits_{s \le T} W_s > 0\right]}{P\left[\min\limits_{s \le T} W_s > 0\right]}$$

$$= \frac{P\left[W_t \in A ; \min\limits_{s \le t} W_s > 0 ; \min\limits_{t \le s \le T} W_s > 0\right]}{P\left[\min\limits_{s \le T} W_s > 0\right]}$$

$$= \frac{\int dy\, P_{x;s}\left[\min\limits_{s \le t} W_s > 0 ; W_t = y\right] P_{y;t}\left[\min\limits_{t \le s \le T} W_s > 0\right]}{P_X\left[\min\limits_{s \le t} W_s > 0\right]}$$

$$\left\{ \min_{s \le T} W_s > 0 \right\} = \left\{ \tau_0 > T \right\}$$

$$\mathbb{1}\{\tau > T\} = \lim_{\varepsilon \downarrow 0} e^{-\frac{1}{\varepsilon} \int_t^T \chi(W_s) ds}$$

Therefore:

$$\zeta(x,t,T) \equiv \mathbb{P}_{xt}\left[\tau_0 > T \right]$$

The conditioned process satisfies

$$dX_t = dW_t + \frac{\zeta_x(x,t;T)}{\zeta(x,t,T)} \cdot dt$$

To calculate $\zeta(x,t;T)$ as $T \to \infty$ and its logarithmic derivative

We just ~~say~~ write

$$\frac{\partial \zeta}{\partial t} + \frac{1}{2} \zeta_{xx} = 0$$

$$\psi = \ln \zeta$$

$$\frac{\partial \psi}{\partial t} + \frac{1}{2}\left(\psi_{xx} + \psi_x^2\right) = 0$$

$$\rho = \psi_x$$

$$\frac{\partial \rho}{\partial t} + \frac{1}{2}\left(\rho_{xx} + 2\rho\rho_x\right) = 0$$

If stationary solution exists

$$\rho_{xx} + 2\rho\rho_x = 0$$

$$\rho_{xx} + (\rho^2)_x = 0$$

~~Also~~ thus:

$$\rho_x + \rho^2 = c$$

Since drift should vanish far away from $x=0$ we have $c=0$.

$$\rho_x = -\rho^2$$

$$\frac{\rho_x}{\rho^2} = -1 \qquad \frac{1}{\rho} = -x$$

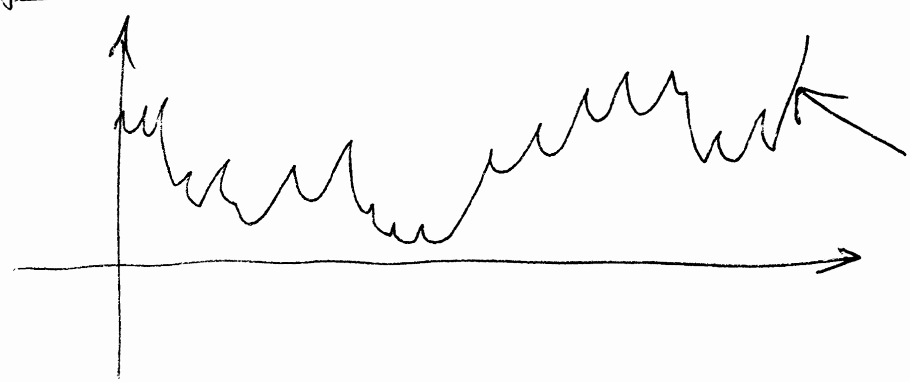
$$\rho = \frac{1}{x}$$

$$\boxed{dX_t = dW_t + \frac{1}{X_t}\,dt}$$

Recall that Bessel process satisfy:

$$dX_t = dW_t + \frac{n-1}{2X_t}\,dt$$

<u>Conclusion</u>: Brownian Motion conditioned on never hitting $w=0$ is distributed like a <u>Bessel-3 process</u> (Pitman-Yor, 1970's?)



radial part of a 3D BM.

what about conditioning an Orstein-Uhlenbeck to remain positive for all times?

O-U.

$$dX_t = dW_t - kX_t\, dt$$

New conditional process

$$dX_t = dW_t - kX_t + \frac{\zeta_x}{\zeta}\, dt$$

$$\frac{\partial \zeta}{\partial t} + \frac{1}{2}\zeta_{xx} - k\zeta_x = 0$$

"Equilibrium" solution

$$\frac{1}{2}\zeta_{xx} = k\zeta_x$$

$$\frac{\zeta_{xx}}{\zeta_x} = 2k$$

$$\ln \zeta_x = 2kx$$

$$\zeta_x = \gamma_1 e^{2kx}$$

$$\zeta = \gamma_1 \frac{e^{2kx}}{2k} + \gamma_2$$

$$\frac{\zeta_x}{\zeta} = \frac{\gamma_1 e^{2kx}}{\gamma_1 \frac{e^{2kx}}{2k} + \gamma_2}$$

$$= \frac{2k\, e^{2kx}}{e^{2kx} - \gamma_3}$$

$$= \frac{2k}{1 - \gamma_3 e^{-2kx}}$$

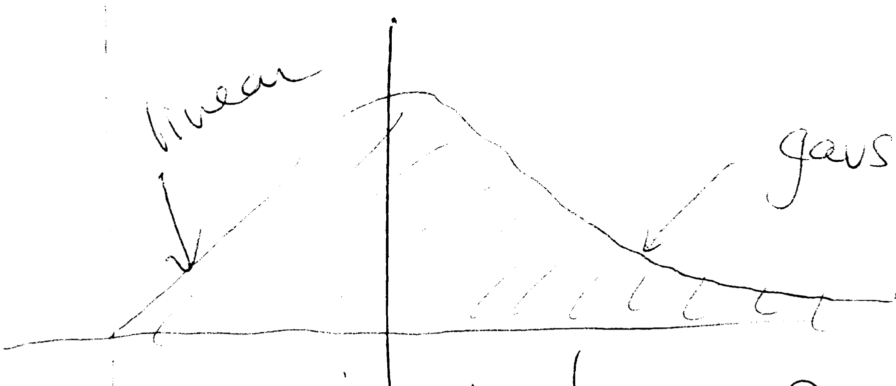
$\frac{\zeta_x}{\zeta}$ ~~It~~ needs to vanish at x=0 to produce infinite drift.

<u>Conclusion</u>: the conditioned process is

$$\boxed{dX_t = dW_t - kX_t + \frac{2k}{1-e^{-2kX_t}}\,dt}$$

It can be shown that this process has invariant measure

$$\pi(x) = \frac{1}{Z}\, e^{-\frac{kx^2}{2}} \left(e^{2kx}-1\right)^+$$



The mean is not zero.