Stochastic Calculus, Lecture 11

Discounting, Conditioning and Forward Measures

Consider the problem of computing the expected value

$$I = E\left\{ e^{-\int_0^T V(X_t,t)\,dt} F(X_T) \right\}$$

where

$$dX_t^i = \sum_{j=1}^{m} \sigma_j^i(X_t,t)\cdot dW_j + \mu^i(X_t,t)dt$$

$$i = 1, \dots, n$$

is a multi-dimensional diffusion.

The Feynman-Kac formalism give

$$\phi(x,t) = E\left\{ e^{-\int_t^T V(X_s,s)\,ds} F(X_T) \,\middle|\, X_t = x \right\}$$

as the solution of the boundary-value problem

$$\begin{cases} \dfrac{\partial \phi}{\partial t} + \dfrac{1}{2}\sum_{ij=1}^{n} a^{ij} \dfrac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_i \mu^i \dfrac{\partial \phi}{\partial x_i} - V\phi = 0 \\ \qquad t < T \\ \phi(\underline{x}, t=T) = F(x) \end{cases}$$

<u>Example</u>: $dX = \sigma dW + \kappa(\theta - X)\, dt$

$$\mathbb{E}\left[e^{-\int_0^T X_t dt} (X_T - K)^+ \right]$$

represents the value of a <u>call option on an interest rate</u> X_T.

Associated PDE

$$\frac{\partial \phi}{\partial t} + \kappa(\theta - x)\frac{\partial \phi}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 \phi}{\partial x^2} - x\phi = 0$$

$$\phi\Big|_{t=T} = (x-K)^+$$

We have an alternative way of thinking about I, namely

$$I = E\left(e^{-\int_0^T V\,ds}\right) \cdot \frac{\mathbb{E}\left\{e^{-\int_0^T V\,ds} F(X_T)\right\}}{\mathbb{E}\left\{e^{-\int_0^T V\,ds}\right\}}$$

$$= E\left(e^{-\int_0^T V\,ds}\right) \cdot \widetilde{\mathbb{E}}\left\{F(X_T)\right\},$$

where we define.

$$\widetilde{E}\left[\Phi(X_s, s \le T)\right] = \frac{E\left\{e^{-\int_0^T V\,ds} \Phi(X_s, s \le T)\right\}}{E\left\{e^{-\int_0^T V}\right\}}.$$

Clearly $\widetilde{E}(\ )$ is a ~~PDF~~ probability.

Let us study $\widetilde{E}(\ )$ in detail.

$$\cancel{(i)\ \widetilde{E}\left[\Phi(X_t)\right] = \frac{\widetilde{E}\left[e^{-\int_0^T V} \phi(X_t)\right]}{\widetilde{E}\left[e^{-\int_0^T V}\right]}}$$

Is $\tilde{E}\{\ \}$ associated with a diffusion process?

If $s \le t$

$$\tilde{E}[F(X_t)\,\Phi(X_s)] =$$

$$E\left[e^{-\int_0^T V(X_u)\,du} F(X_t)\,\phi(X_s)\right] \Big/ E\left(e^{-\int_0^T V}\right)$$

$$= \mathbb{E}\left[e^{-\int_0^s V\,du}\,\phi(X_s)\cdot e^{-\int_s^T V\,du} F(X_t)\right] \Big/ Z$$

$Z = Ee^{-\int_0^T V}$

$$= \frac{\mathbb{E}\left[e^{-\int_0^s V}\,\phi(X_s)\,\mathbb{E}\left[e^{-\int_s^T V\,du} F(X_t)\,\middle|\,X_s\right]\right]}{Z_0^T}$$

$$= \frac{E\left(e^{-\int_0^s V}\,\phi(X_s)\,E\left[e^{-\int_s^T V}\,\middle|\,X_s\right]\cdot \dfrac{E\left[e^{-\int_s^T V} F\,\middle|\,X_s\right]}{E[\ |X_s]}\right)}{Z_0^T}$$

⑤

$$= \frac{E\left\{ e^{-\int_0^T V} \phi(X_s) \dfrac{E\left[e^{-\int_s^T V} F \mid X_s\right]}{E\left[e^{-\int_s^T V} \mid X_s\right]} \right\}}{E\, e^{-\int_0^T V}}$$

$$= \tilde{\mathbb{E}} \left\{ \phi(X_s) \frac{E\left[e^{-\int_s^T V} F \mid X_s\right]}{E\left[e^{-\int_s^T V} \mid X_s\right]} \right\}$$

$$\therefore \quad \tilde{\mathbb{E}}\left[F \mid X_u, u \le s\right] = \frac{\mathbb{E}\left[e^{-\int_s^T V} F \mid X_s\right]}{E\left[e^{-\int_s^T V} \mid X_s\right]}$$

It follows from this ~~that~~ if X_t is Markov (a diffusion) so is the ~~[illegible]~~ process associated with $\tilde{E}(\ )$.

$$\tilde{E}[F \mid X_t = x] = \frac{E\left[e^{-\int_t^T v} F \mid X_t = x\right]}{E\left[e^{-\int_t^T v} \mid X_t = x\right]}$$

Let $F = F(X_T)$.

$$\phi(x,t) = \tilde{E}[F(X_T) \mid X_t = x]$$

$$\psi(x,t) = E\left[e^{-\int_t^T v} F(X_T) \mid X_t = x\right]$$

$$\zeta(x,t) = E\left[e^{-\int_t^T v} \mid X_t = x\right]$$

$$\phi(x,t)\,\zeta(x,t) = \psi(x,t)$$

$$\phi_t \zeta + \phi \zeta_t = \psi_t$$

$$\phi_t \zeta = \psi_t - \phi \zeta_t$$

$$\phi_t \zeta = -\mathcal{L}\psi + V\psi - \phi(-\mathcal{L}\zeta + V\zeta)$$

$$= -\mathcal{L}\psi + V\psi + \phi\mathcal{L}(\zeta) - V(\zeta\phi)$$

$$\cancel{= -\mathcal{L}\psi + \mathcal{L}(\phi\zeta)}$$

$$\cancel{= -\mathcal{L}\psi}$$

$$= -\mathcal{L}\psi + \phi\mathcal{L}\zeta$$

$$= -\mathcal{L}(\phi\zeta) + \phi\mathcal{L}\zeta$$

$$= -\left[\mathcal{L}\phi\,\zeta + \mathcal{L}(\zeta)\phi + a\cdot\nabla\phi\,\nabla\zeta\right]$$
$$+\ \phi\mathcal{L}\zeta$$

$$\phi_t \zeta = -\mathcal{L}\phi\,\zeta - a\nabla\phi\,\nabla\zeta$$

<u>drift</u> in the corresponding SDE.

<u>Example</u> :

$$dX_t = k(\theta - x)\,dt + \sigma\, dW_t$$

$$\zeta(x,t) = E\left\{ e^{-\int_t^T X_s ds} \,\middle|\, X_t = x \right\}$$

$$\frac{\partial \zeta}{\partial t} + k(\theta - x)\frac{\partial \zeta}{\partial x} + \frac{1}{2}\sigma^2 \frac{\partial^2 \zeta}{\partial x^2} - x\zeta = 0$$

$$\zeta = e^{ax+b}$$

$$\dot{a}x + \dot{b} + k(\theta - x)a + \frac{1}{2}\sigma^2 a^2 - x = 0$$

$$\begin{cases} \dot{a} - ak - 1 = 0 \\ \dot{b} + k\theta a + \frac{1}{2}\sigma^2 a^2 = 0. \end{cases}$$

$$a = -\frac{1 - e^{-k(T-t)}}{k}$$

$$\frac{\zeta_x}{\zeta} = a = -\frac{1-e^{-k(T-t)}}{k}$$

$$\boxed{dY_t = \kappa(\theta - Y_t)\,dt - \frac{\sigma^2}{K}\left(1-e^{-k(T-t)}\right)dt + \sigma\, dW_t}$$

$$E\left[e^{-\int_0^T X_s ds}(X_T-K)^+\right] =$$

$$= \zeta(x,T)\cdot E\left[(Y_T-K)^+\right]$$

$$= e^{-rT} E\left[(Y_T-K)^+\right].$$

Another example corresponds to an option on a zero-

$$\begin{cases} \phi_t + \mathcal{L}\phi + a\,\dfrac{\nabla\zeta}{\zeta}\cdot\nabla\phi = 0 & t<T \\ \phi\big|_{t=T} = F(x) \end{cases}$$

Interpretation: the probability $\tilde{P}$ associated with $\tilde{E}(\ )$ corresponds to a new diffusion Y_t given by

$$\boxed{dY_t^i = \sum_{k=1}^{m} \sigma_k^i(Y_t,t)\cdot dW_t^k + \mu^i(Y_t,t)\cdot dt + \sum_{j=1}^{n} a^{ij}(y,t)\,\frac{\partial \zeta(y,t)}{\partial y_j}\,\frac{1}{\zeta(y,t)}\,dt}$$

Thus the change of measure corresponds to a change of coupon bond

$$Z_{t,T_1} = E\left\{ e^{-\int_t^{T_1} X_s\,ds} \;\middle|\; X_t = x \right\}$$

Call option on ZCB maturing @ time T_1

$$E\left\{ e^{-\int_0^T X_s\,ds} \max\left(Z_{T,T_1} - K, 0 \right) \right\}$$

$$= E\left[e^{-\int_0^T X_s\,ds} \right] \tilde{E}\left[\max\left(Z_{T,T_1} - K, 0 \right) \right]$$

$$= Z_{0,T} \cdot \tilde{E}\left[\max\left(Z_{T,T_1} - K, 0 \right) \right]$$

(i) Z_{T,T_1} is log-normal

~~(ii) Volatility is~~ $\cancel{\sigma^2_{T,T_1} = \sigma^2 \cdot \frac{1 - e^{-K(T_1 - T)}}{K}}$

Let us calculate its (Black-Scholes) volatility:

$$Z_{t,T} = e^{-\left(\frac{1-e^{-k(T-t)}}{k}\right) X_t + \cdots}$$

$$X_t = X_0 e^{-kt} + (1-e^{-kt})\theta + \sigma \int_0^t e^{-k(t-s)} dW_s$$

$$Z_{t,T} = e^{-\sigma\left(\frac{1-e^{-k(T-t)}}{k}\right)\int_0^t e^{-k(t-s)} dW_s + \cdots}$$

$\therefore$

$$\sigma^2(Z_{\cancel{t},T_1}) \cdot T = \sigma^2 \left(\frac{1-e^{-k(T_1-T)}}{k}\right)^2 \cdot \frac{1-e^{-2kT}}{2k}$$

$$\sigma^{Z}(Z_{T,T_1}) = \sigma \sqrt{\frac{1-e^{-2KT}}{2KT} \cdot \frac{1-e^{-K(T_1-T)}}{K}}$$

It follows that

$$C(Z_{0,T}, Z_{0,T_1}, \sigma, K) =$$

$$= Z_{0T} \; BSC\left\{ Z_{0,T_1}, T, K, 0, 0, \sigma(T, T_1) \right\}$$

$$= Z_{0,T}\left\{ \frac{Z_{0,T_1}}{Z_{0,T}} N[d_1] - K\,N(d_2) \right\}$$

$$\boxed{C = Z_{0,T_1} N(d_1) - K Z_{0T} N(d_2)}$$

<u>Conclusion:</u> If the interest rate is <u>Normal</u> (eg OU), the calculation of prices of options on zero-coupon bonds can be done using the standard BS formula.