Stochastic Calculus: Lecture 10

1. Stochastic Differential Equations

So far, we have taken for granted that we could construct processes such that

$$dX_t = \sigma(X_t,t)\,dZ_t + \mu(X_t,t)\,dt,$$

where $\sigma(x,t)$, $\mu(x,t)$ are given functions. We have seen a few examples where the construction was explicit.

1. Lognormal Diffusion

$$\boxed{dS_t = \sigma S_t \cdot dW_t + \mu S_t\,dt}$$

$$d\ln S_t = \frac{1}{S_t}\cdot dS_t - \frac{1}{2}\frac{1}{S_t^2}\,dS_t^2$$

$$= \sigma\,dW_t + \mu\,dt - \frac{1}{2}\sigma^2\,dt$$

$$\ln S_t = \ln S_0 + \sigma W_t + \mu t - \frac{1}{2}\sigma^2 t$$

$$S_t = S_0 e^{\sigma W_t - \frac{1}{2}\sigma^2 t + \mu t}$$

2. <u>Ornstein-Uhlenbeck</u>

$$\boxed{dX_t = \sigma\, dW_t + \kappa(\theta - X_t)\,dt}$$

use "integrating factor" $X_t e^{kt} = Y_t$

$$d(X_t e^{kt}) = dX_t e^{kt} + k X_t e^{kt} dt$$
$$= (\sigma\, dW_t + k\theta\, dt) e^{kt}$$

$$X_t e^{kt} - X_0 = \sigma \int_0^t e^{ks} dW_s + k\theta \left(\frac{e^{kt} - 1}{k}\right)$$

$$X_t = e^{-kt} X_0 + \theta(1 - e^{-kt}) + \sigma \int_0^t e^{-k(t-s)} dW_s$$

Once again, we have an <u>explicit solution</u> (or construction).

3. <u>Bessel process</u>

$$dX_t = \cancel{\#}\, dW_t + \frac{n-1}{2X_t} dt$$

We can construct the Bessel process if $n \in \mathbb{N}$ $n \geqslant 2$.

In fact set

$$X_t = \sqrt{\sum_i (W_i(t))^2}$$

$$f(\xi) = (\xi_1^2 + \cdots + \xi_n^2)^{1/2} = |\xi|$$

$$\nabla_i f(\xi) = \frac{\xi_i}{|\xi|}$$

$$\nabla_i \nabla_j f = -\frac{\xi_i \xi_j}{|\xi|^3} + \frac{\delta_{ij}}{|\xi|}$$

$$= \frac{1}{|\xi|}\left(\delta_{ij} - \frac{\xi_i \xi_j}{|\xi|^2}\right)$$

$$dX_t = \frac{\sum_i W_i \cdot dW_i}{X_t} + \frac{1}{2} \sum_{ij} \frac{\partial^2 f}{\partial x_i \partial y} dW_i dW_j$$

$$= \frac{\sum_i w_i \cdot dW_i}{\sqrt{\sum w_j^2}} + \frac{1}{2}\frac{1}{X_t}\sum_{ij}\left(\delta_{ij} - \frac{w_i w_j}{|W|^2}\right) dW_i dW_j$$

$$= \frac{\sum_i w_i\, dW_i}{\sqrt{\sum_j W_j}} + \frac{1}{2X}\sum_{ij}\left(\delta_{ij} - \frac{w_i w_j}{|W|^2}\right)\delta_{ij}\, dt$$

$$= \frac{\sum_i w_i\, dW_i}{\sqrt{\sum_j W_j}} + \frac{1}{2}\frac{(n-1)}{X_t}\, dt$$

But $dZ_t :=$

$$= \frac{\sum_i w_i \cdot dW_i}{\sqrt{\sum_i w_i^2}}$$

defines a BM, since

$$d\left(e^{\lambda Z_t - \frac{1}{2}\lambda^2 t}\right)$$

$$= \lambda e^{\lambda Z_t} \cdot dZ_t + \frac{1}{2}\lambda^2 e^{\lambda Z_t} dt - \frac{1}{2}\lambda^2 \cdot e^{\lambda Z_t}$$

$$= e^{\lambda Z_t} \cdot \left[\left(\lambda\, dZ_t + \frac{1}{2}\lambda^2 dt\right) - \frac{1}{2}\lambda^2 dt\right]$$

$$= e^{\lambda Z_t} \lambda \cdot \frac{\sum W_i\, dW_i}{\sqrt{\sum W_i^2}}$$

$$E\left[e^{\lambda Z_t - \frac{1}{2}\lambda^2 t} \,\middle|\, W_u,\ u \le s\right]$$

$$e^{\lambda Z_s - \frac{1}{2}\lambda^2 s}$$

$$\therefore\ \mathbb{E}\left(e^{\lambda(Z_t - Z_s)} \,\middle|\, W_u\ u \le s\right) = e^{\frac{\lambda^2}{2}(t-s)}.$$

This shows that Z is

Gaussian with indept increments
hence a BM. Also $X_t =$
$\left(\sum W_i(t)^2\right)^{1/2}$ is non-anticipative
with respect to the BMs $(W_1 \cdots W_n)$.
here with respect to Z.
We "constructed" a solution of

$$dX_t = dW + \frac{n-1}{2X_t}\,dt.$$

What happens in general? (Eg $n \notin \mathbb{N}$)

We need to show that given
nice $\sigma(x,t)$, $\mu(x,t)$, we
can find a process X_t such
that

$$(*) \quad X_t = X_0 + \int_0^t \sigma(X_s, s)\, dW_s + \int_0^t \mu(X_s, s)\, ds$$

<u>Pathwise solution</u>: Given a Brownian motion prob on $\mathcal{C}(0,T)$, we can find $X_t = \Phi(W_s, s\le t; t)$ such ~~that~~ $(*)$ holds.

<u>Theorem</u>: If $\sigma(x,t)$, $\mu(x,t)$ are bounded and Lipschitz continuous there exists a unique solution which can be constructed by iteration.

$|\sigma(x,t)| \le M$, $|\mu(x,t)| \le M$

$$\begin{cases} |\sigma(x,t) - \sigma(x',t)| \leq K |x - x'| \\ |\mu(x,t) - \mu(x',t)| \leq K' |x - x'| \end{cases}$$

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<u>Proof</u>:

$$X_t^{(n)} = \gamma + \int_0^t \sigma(X_s^{(n-1)}, s) \cdot dW_s + \int_0^t \mu(X_s^{(n-1)}, s)\, ds$$

Construct $X_t^{(n)}$ <u>iteratively</u>.

$$X_t^{(n)} - X_t^{(n-1)} =$$

$$= \int_0^t \left(\sigma(X_s^{(n-1)}, s) - \sigma(X_s^{(n-2)}, s)\right) \cdot dW_s + \int_0^t \left(\mu(X_s^{(n-1)}, s) - \mu(X_s^{(n-2)}, s)\right) ds$$

$$\sup_{t \le T} \left| X_t^{(n)} - X_t^{(n-1)} \right| \le$$

$$\le \sup_{t \le T} \left| \int_0^t \left(\sigma(X_s^{(n-1)}, s) - \sigma(X_{s}^{(n-2)}, s) \right) . dw_s \right|$$

$$+ \sup_{t \le T} \left| \int_0^t \left(\mu(X_s^{(n-1)}, s) - \mu(X_{s}^{(n-2)}, s) \right) ds \right|$$

Lemma: $(\xi_n)_{n \le N}$ is a Martingale

$$\boxed{E\left(\max_{n \le N} |\xi_n|^2 \right) \le E\, \xi_N^2}$$

Proof: See Varadhan's notes.
(Kolmogorov - Doob's ineq.)

$$\mathbb{E}\left[\sup_{t\le T}\left|X_t^{(n)}-X_t^{(n-1)}\right|^2\right] \le$$

$$2\,\mathbb{E}\left[\int_0^T\left(\sigma^{(n)}-\sigma^{(n-1)}\right)^2 ds\right] +$$

$$2\,\mathbb{E}\left|\int_0^T\left(\mu^{(n)}-\mu^{(n-1)}\right)ds\right|^2$$

$$\le 2TK^2\,\mathbb{E}\left[\sup_{t\le T}\left|X_t^{(n-1)}-X_t^{(n-2)}\right|^2\right]$$

$$+\,2T^2K^2\,\mathbb{E}\left[\sup_{t\le T}\left|X_t^{(n-1)}-X_t^{(n-2)}\right|^2\right]$$

$$\mathbb{E}\left\{\sup_{t\le T}\left|X_t^{(n)}-X_t^{(n-1)}\right|^2\right\}\le$$

$$4K^2\max(T,T^2)\cdot\,\mathbb{E}\left[\sup_{t\le T}\left|X_t^{(n-1)}-X_t^{(n-2)}\right|^2\right]$$

If we choose T^* such ~~that~~ that

$$4k^2 T^* V T^{*2} \le \alpha < 1$$

$$E\left\{\sup_{t\le T}\left|X_t^{(n)} - X_t^{(n-1)}\right|^2\right\} < \alpha E\left\{\sup_{t\le T}\left|X_t^{(n-1)} - X_t^{(n-2)}\right|^2\right\}$$

$\therefore$ $$\boxed{E\left\{\sup_{t\le T}\left|X_t^{(n)} - X_t^{(n-1)}\right|^2\right\} \le C\alpha^n}$$

$$P\left[\sup_{t\le T}\left|X_t^{(n)} - X_t^{(n-1)}\right| > \varepsilon\right] \le \frac{C\alpha^n}{\varepsilon^2}$$

$\therefore$

$$P\left[\sup_{t\le T}\left|X_t^{(n)} - X_t^{(n-1)}\right| > \beta^{n/2}\right] \le C\left(\frac{\alpha}{\beta}\right)^n \quad (\alpha < \beta < 1)$$

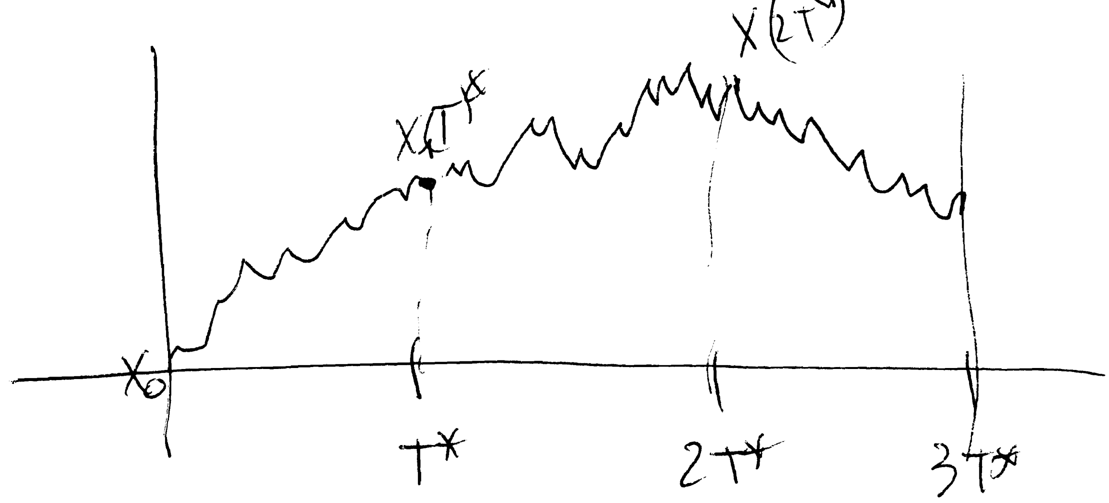
By Borel-Cantelli's Lemma

$$\boxed{P\left\{ \sup_{t \le T} \left| X_t^{(n)} - X_t^{(n-1)} \right| > \beta^{n/2} \text{; i.o.} \right\} = 1}$$

$\therefore$ Uniform convergence of $X_t^{(n)}$ to a continuous fun with probability 1.

Uniqueness follows from applying the previous argument to $D(t) = X(t) - Y(t)$ which are two solutions.

Note: This holds for $T < T^*$, However the range can be extended by gluing together solutions.



Note: This means that we can approximate the SDE discretely as well

$$X_{n\Delta T} = X_{(n-1)\Delta T} + \sigma\left(X_{(n-1)\Delta T}, (n-1)\Delta T\right) \cdot N_n \sqrt{\Delta T} + \mu\left(X_{(n-1)\Delta T}, (n-1)\Delta T\right) \Delta T$$

("Forward-Euler" method).

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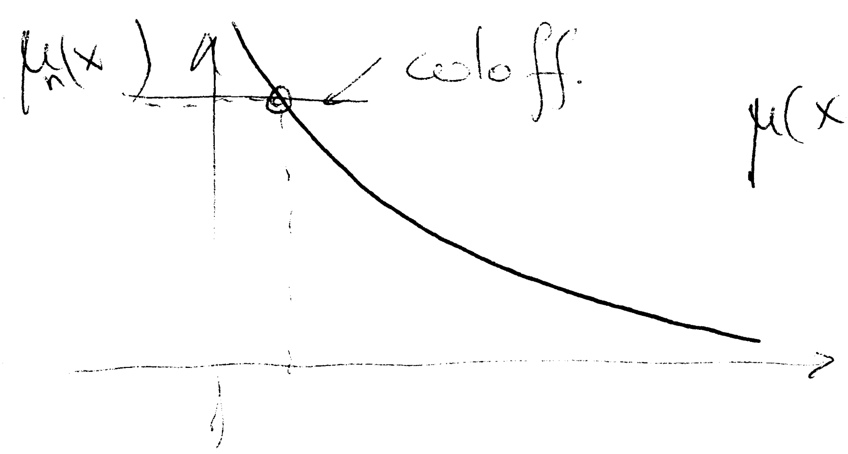
__Application__: Fractional Bessel process

Give:

$$dX_t = dW_t + \frac{(n-1)}{2X_t}\,dt$$

$n \geq 2 \quad n \in \mathbb{R}$.



$$\mu(x) = \frac{n-1}{2X}$$

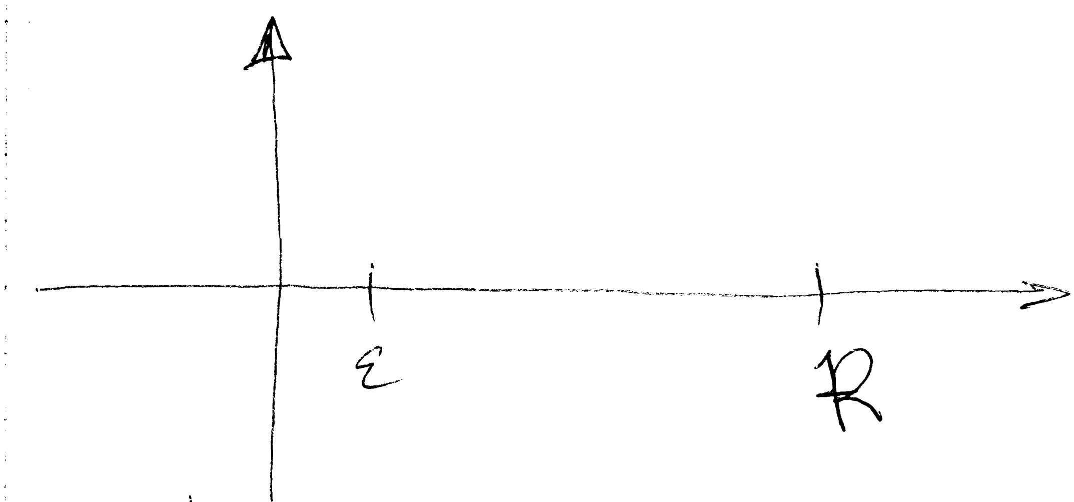
$\mu(x)$ is not bounded or uniformly Lipschitz

$$\hat{\mu}(X) = \min\left(M, \frac{n-1}{2x}\right).$$

A unique solution exists. For this solution met coincide with Bessel until the first passage time of

$\mu(X_t) = M$

$$\frac{n-1}{2X} = M \quad \therefore \quad X = \frac{(n-1)}{2M} \equiv \varepsilon$$



$\varepsilon \ll 1$

$$P_x\left[X(\tau_{\varepsilon R}) = \varepsilon \right] = \phi(x)$$

$$\begin{cases} \mathcal{L}\phi = 0 & \varepsilon < x < R \\ \phi(\varepsilon) = 1 \\ \phi(R) = 0 . \end{cases}$$

$$\frac{1}{2}\phi'' + \frac{1}{2}\,\frac{\phi'(n-1)}{x} = 0$$

$$\frac{\phi''}{\phi'} = -\frac{n-1}{x}$$

$$\phi' = \frac{C_1}{x^{n-1}}$$

$$\Phi = C_0 + \frac{C_1}{x^{n-2}}$$

$$\boxed{\phi(x) = \frac{\frac{1}{R^{n-2}} - \frac{1}{x^{n-2}}}{\frac{1}{R^{n-2}} - \frac{1}{\varepsilon^{n-2}}}}$$

~~P[X ~~zero~~ R bef X hits zero]~~

=

$P[\,x$ hits zero bef. $R\,] \leq$

$P[\,X$ hits ε bef $R\,] =$

$$\frac{\frac{1}{R^{n-2}} - \frac{1}{x^{n-2}}}{\frac{1}{R^{n-2}} - \frac{1}{\varepsilon^{n-2}}} = \frac{\frac{1}{x^{n-2}} - \frac{1}{R^{n-2}}}{\frac{1}{\varepsilon^{n-2}} - \frac{1}{R^{n-2}}}$$

$\lim_{\varepsilon \downarrow 0} P[\,X$ hits ε bef. $R\,] = 0$

$$\boxed{P[\,X \text{ hits zero before } X \text{ hits } R\,] = 0}$$

The Bessel process exists for $n \geq 2$. If $n < 2$ zero is hit with finite prob.

The CIR process (Feller process)

$$dX_t = \alpha\sqrt{X_t}\cdot dZ_t + \kappa(\theta - X_t)\,dt$$

Locally, coeffs are bounded and Lipschitz, so a unique pathwise solution exists until X_t hits zero.

$P_x[\,X \text{ hits } \varepsilon \text{ before } X \text{ hits } R\,]$

$= \phi(x)$

$$\begin{cases} \frac{\alpha^2}{2} x\,\phi_{xx} + \kappa(\theta - x)\,\phi_x = 0 \\ \phi(\varepsilon) = 1 \qquad \phi(R) = 0 \end{cases}$$

$$\frac{\phi_{xx}}{\phi_x} = -\frac{2k(\theta - x)}{\alpha^2 x}$$

$$\ln \phi_x = -\frac{2k\theta}{\alpha^2}\ln(x) + \frac{2kx}{\alpha^2} + C$$

$$\phi_x = C x^{-\frac{2k\theta}{\alpha^2}} e^{\frac{2kx}{\alpha^2}}$$

$$\boxed{\Phi(x) = \frac{\int_x^R e^{+\frac{2kx}{\alpha^2}} x^{-\frac{2k\theta}{\alpha^2}} dx}{\int_\varepsilon^1 e^{\frac{2kx}{\alpha^2}} x^{-\frac{2k\theta}{\alpha^2}} dx}}$$

If $\frac{2k\theta}{\alpha^2} \geq 1$, the denominator diverges and

$$\lim_{\varepsilon \downarrow 0} P[\, X \text{ hits } \varepsilon \text{ before } X \text{ hits } R\,] = 0.$$

<u>Intuition</u>: If

$$y_t = \sum_i X_i^2(t)$$

where X_i are OU

$$dX_i = -\kappa X_t\,dt + \sigma\,dW_t$$

$\therefore$

$$dy_t = 2\sum_i X_i\,dX_i + \sum_i dX_i^2$$

$$= 2\sum_i X_i\left(-\kappa X_i\,dt + \sigma\,dW\right)$$

$$+ \sum_i \sigma^2\,dt$$

$$= -2\kappa\, y_t\,dt + n\sigma^2\,dt$$

$$+ 2\sigma\sum_i X_i\,dW_i$$

$$dy_t = -2k y_t\,dt + n\sigma^2 dt + 2\sigma\sqrt{y_t}\;\frac{\sum_i X_i\,dW_i}{\sqrt{\sum_i X_i^2}}$$

$$= -2k y_t\,dt + n\sigma^2 dt + + 2\sigma\sqrt{y_t}\cdot dZ_t$$

$$\tilde{K} = 2K \qquad \tilde{\theta} = \frac{n\sigma^2}{\tilde{K}} = \frac{n\sigma^2}{2K}$$

$$\tilde{\alpha} = 2\sigma$$

$$\frac{2\tilde{K}\tilde{\theta}}{\tilde{\alpha}^2} = \frac{2(2k)\,\frac{n\sigma^2}{2K}}{4\sigma^2} = \frac{n}{2}$$

This, n is the dimension, or

number of degrees of freedom.

$$\boxed{\frac{2\breve{K}\breve{\theta}}{\alpha^2} \geq 1 \iff n \geq 2}$$

The study of $\frac{2K\theta}{\alpha^2} < 1$ is beyond the scope of this course.

Example:

$$dy_t = \alpha\sqrt{y_t}\,dW_t + \mu\,dt$$

$\boxed{\alpha, \mu \text{ constant.}}$

Exists for all $\{|y| > \varepsilon\}$.

$$P_x[\tau_\varepsilon < \tau_R] = \phi(x)$$

$$\frac{1}{2}\alpha^2 x \phi_{xx} + \mu \phi_x = 0$$

$$\frac{\phi_{xx}}{\phi_x} = -\frac{2\mu}{\alpha^2}\frac{1}{x}$$

$$\ln \phi_x = c - \frac{2\mu}{\alpha^2}\ln x$$

$$\phi_x = c\, x^{-\frac{2\mu}{\alpha^2}}$$

$$\Phi = c\, x^{1-\frac{2\mu}{\alpha^2}} \qquad \frac{2\mu}{\alpha^2} \neq 1$$

$$\phi_\varepsilon(x) = \frac{R^{1-\frac{2\mu}{\alpha^2}} - x^{1-\frac{2\mu}{\alpha^2}}}{R^{1-\frac{2\mu}{\alpha^2}} - \varepsilon^{1-\frac{2\mu}{\alpha^2}}}$$

if $\frac{2\mu}{\alpha^2} > 1$ then

$\lim_{\varepsilon \downarrow 0} \phi_\varepsilon(x) = 0$ and ~~path~~ is well-defined.

The differnce is in the behavior at infinity.

Fokker - Planck Equations

Let $\pi(x,t,y,T)$ be the fundamental solution of the equation $\frac{\partial}{\partial t} + \mathcal{L} = 0$.

$$\begin{cases} \left(\frac{\partial}{\partial t} + \mathcal{L}_x\right)\left(\pi(x,t;y,T)\right) = 0 \\ \pi(x,t;y,T)\Big|_{t=T} = \delta(x-y) \end{cases}$$