Stochastic Calculus

Lecture 1

1. Stochastic Processes

A r.v. is a numerical (real valued) function defined on our probability space. It is denoted by X.
A random vector, or vector of rv's is

$$(X_1, X_2, \dots X_N)$$

it is specified by the joint probability distribution.

<u>Example</u>: Sequence of independent normal r.v.s with mean ~~zero and var~~ μ and variance σ.

$$f(x_1 \dots x_n) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma_\varepsilon^2}}$$

$$P[X_1 \in A_1, X_2 \in A_2, \dots X_n \in A_n] =$$

$$\int_{A_1}\int \cdots \int_{A_N} f(x_1 \dots x_n)\, dx_1 \dots dx_n .$$

density

~~Gaussian random vectors are defined by a joint PDF~~

~~Let Σ be a matrix~~

~~$\Sigma_{ij} = E\{X_i, X_j$~~

In general, if X is a r.v.

$EX = \mu_X$ is the mean

$EX^2 - (EX)^2 = \sigma_X^2$ is the variance

For a vector, $E X_i = \mu_i$

$$E(X_i - \mu_i)(X_j - \mu_j) = \mathrm{Cov}(X_i, X_j)$$

$\mathrm{Cov}(X_i, X_i) = \mathrm{Variance}(X_i)$

$C_{ij} \geq 0.$ $\cancel{E}$

$$\sum_{ij} C_{ij} \theta_i \theta_j = \sum_{ij} E(\tilde{X}_i \tilde{X}_j) \theta_i \theta_j$$

$$= E \sum_{ij} \tilde{X}_i \tilde{X}_j \theta_i \theta_j$$

$$= E \left| \sum_i \tilde{X}_i \theta_i \right|^2 \geq 0.$$

If $C_{ij}\theta_i\theta_j = 0$ then

$$\sum_i X_i\theta_i = \text{constant with prob } 1$$

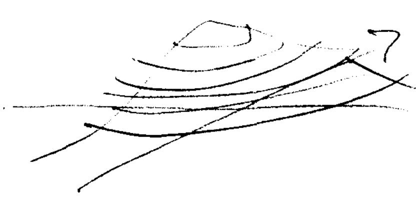
$(X_1 \dots X_n)$ is <u>Gaussian</u> if $\exists\, C, \mu$ such that $\forall (\theta_1 \dots \theta_n)$

$$\sum_i \theta_i X_i \sim N\left(\sum_i \mu_i\theta_i,\ \sum_{ij} C_{ij}\theta_i\theta_j\right)$$

<u>Joint PDF</u>: Assume $C_{ij} > 0$

$$f(x_1 \dots x_n) = \left(\frac{1}{\sqrt{2\pi}}\right)^n (\det C)^{-1/2} \exp\left\{-\frac{1}{2}\sum_{ij} C^{-1}_{ij}(x_i-\mu_i)(x_j-\mu_i)\right\}$$

~~Note~~ Verify the $\int f = 1$.



$\rightarrow$ Moment-generating fn of the fn.

$$E\left[e^{i\sum_j \theta_j X_j}\right] = \psi_X(\theta_1 \dots \theta_n).$$

Prop: If $(X_1 \dots X_n)$ is Gaussian $N(\mu, C)$

$$\boxed{\psi_X(\theta_1 \dots \theta_n) = e^{i\theta\cdot\mu - \frac{1}{2}\theta' \cdot C\cdot\theta}}$$

Note: this does not require C to be invertible.

$X \sim N(\mu, \sigma^2)$ $\quad E\, e^{i\theta X} =$

$$= \int_{-\infty}^{+\infty} e^{i\theta x}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma^2}}$$

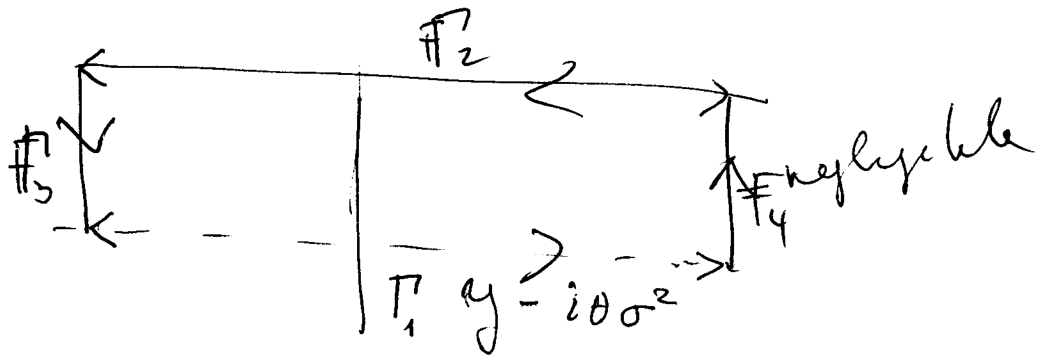
$$= \int_{-\infty}^{+\infty} e^{+i\theta(x-\mu)}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sqrt{2\pi\sigma}}\; e^{i\theta\cdot\mu}$$

$$= \int e^{i\theta y - \frac{1}{2}\frac{y^2}{\sigma^2}} \frac{dy}{\sqrt{2\pi\sigma^2}} \cdot e^{i\theta\cdot\mu}$$

$$= \int e^{-\frac{1}{2\sigma}(y - i\theta\sigma^2)^2} \frac{dy}{\sqrt{2\pi\sigma^2}}\; e^{i\theta\mu - \frac{1}{2}\sigma^2\theta^2}$$

...

$\theta > 0$



$$\int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4} = 0$$

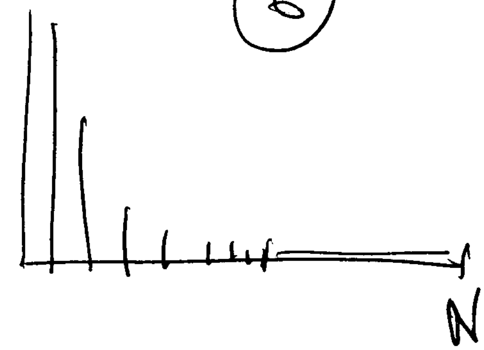
$$\int_{-\infty - i\sigma^2\theta}^{+\infty - i\theta\sigma^2} e^{-\frac{y^2}{2\sigma^2}} \frac{dy}{\sqrt{2\pi\sigma^2}} = \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2\sigma^2}} \frac{dy}{\sqrt{2\pi\sigma^2}} = 1.$$

$$\varphi_X(\theta) = e^{i\theta\mu - \frac{1}{2}\theta^2\sigma^2}.$$

<u>For a vector</u>: take $\mu = 0$ for simplicity

C = covariance matrix

$$C_{ij} = \sum_{k=1}^{m} \lambda_k V_i^{(k)} V_j^{(k)}$$



$$\cancel{X_k} \qquad Y_k = \sum_i V_i^k X_i$$

<u>Y_k is Gaussian.</u> $\quad E\, Y_k Y_{k'} =$

$$E \sum_{ij} V_i^k X_i V_j^{k'} X_j \;.$$

$$\sum_{ij} V_i^k C_{ij} V_j^{k'} = \sum_i V_i^k \lambda^{k'} V_i^{k'}$$

$$= \delta^{kk'} \lambda^{k'}.$$

$$\sigma_k^2 = \lambda_k$$

$$E\, e^{\sum i \theta_k Y_k} = \prod_k e^{-\frac{1}{2}\theta_k^2 \sigma_k^2}$$

$$E\, e^{i\theta_j X_j} = E\, e^{i\theta_j \sum_k V_j^k Y_k}$$

. . .

$$= E\, e^{i\sum_k \left(\sum_j \theta_j V_j^k\right) Y_k}$$

$$= e^{-\frac{1}{2}\sum_k \left|\sum_j \theta_j V_j^k\right|^2 \lambda^k}$$

$$= e^{-\frac{1}{2} C_{ij}\,\theta_i \theta_j}$$

This defn of Gaussianity admits the degenerate case.

This technique of analysis of variance is extremely important in analyzing MV data — whether the vector is Gaussian or not.

Example: $(R_1^t \ldots R_N^t)$ daily returns on a group of N stocks. We can view R_i as R.V. and t as the realization parameter.

$$\begin{cases} \hat{\mu}_i = \dfrac{1}{\nu} \displaystyle\sum_{t=1}^{\nu} R_i^t \\[2ex] \hat{\sigma}_i^2 = \dfrac{1}{\nu-1} \displaystyle\sum_{t=1}^{\nu} \left(R_i^t - \hat{\mu}_i\right)^2 \end{cases}$$

$$\hat{C}_{ij} = \cancel{\frac{1}{\sigma_i \sigma_j}} \cdot \frac{1}{\nu-1} \sum_{t=1}^{\nu} (R_i^t - \mu_i)(R_j^t - \mu_j)$$

Usually, we add a small diagonal noise to prevent degeneracy

$$\hat{C}^{\varepsilon}_{ij} = \hat{C}_{ij} + \varepsilon \delta_{ij}$$

$$\hat{R}^{\varepsilon}_{ij} = \frac{\hat{C}^{\varepsilon}_{ij}}{\sqrt{\hat{C}^{\varepsilon}_{ii} \hat{C}^{\varepsilon}_{jj}}}$$

$$\therefore \boxed{\hat{R}^{\varepsilon}_{ij} = \sum_{k=1}^{N} \lambda^k V_i^{(k)} V_j^{(k)}}$$

$$\sum \lambda^k = N$$

. . .

Thus we can use the data to model (9)
returns, or actually the covariance
of return.
This is very important in the
analysis of multivariate data.

A stochastic process is a (random vector), in
which the index is time. (is an
increasing function).

Discrete time:

$$(X_{t_1}, X_{t_2}, X_{t_3}, X_{t_4} \cdots)$$

Continuous time:

$$X_t, \quad t \in (0, T)$$

The definition of a discrete-time
process is the same as a random
vector: we need to specify the
joint df or chf to get them.

probs. etc. Continuous time is more difficult.

~~In practice~~, Often, due to practical considerations we are interested in ~~dis~~ recursive of difference equations.

$$\begin{cases} X_{n+1} = a + bX_n + V_{n+1} & n \geq 1. \\ \{V_n\} \sim N(0, \sigma^2) & \text{i.d.} \end{cases}$$

This is called an AR 1 model

Solving:

$$X_1 = a + bX_0 + V_1$$

$$X_2 = a + bX_1 + V_2$$

$$= a + b(a + bX_0 + V_1) + V_2$$

$$X_3 = \dots$$

. . .