Stochastic Calculus, Fall 2010
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Homework 2

A. Conditional Expectation. Let $B(t), t > 0$ represent Brownian motion, with $B(0) = 0$. Compute

1. $E\left[B((t)|(B(s))^3\right] \quad (s < t)$.
2. $E\left[\int_0^T B(t)dt|B(u), u \ \leq s\right] \quad (s < t < T)$.
3. $E\left[sin(B(t))|B(s)\right] \quad (s < t)$.
4. $E\left[B(t)|min\left\{B(s), s \leq t\right\} > -1 \ \right]$.

B. Brownian Scale Invariance. Let B(t) be Brownian motion.

1. If $\lambda > 0$ then $X_\lambda(t) \ = \lambda B(t/\lambda^2)$ is a Brownian motion.
2. $X(t) = tB(1/t), \quad t > 0$ is a Brownian motion. Verify that $X(0) = 0$. ("Time inversion.")

C. AR processes (i) Let X_n be a discrete time AR(1) process, such that

$$X_{n+1} = \frac{1}{2}X_n + \frac{1}{8}\nu_{n+1}$$

where ν_k are independent, identically distributed $N(0,1)$. Show that X_n has a long-term, or equilibrium distribution for $n \to \infty$. Also, compute the correlation between X_n and X_{m+n} as $n \to \infty$. (ii) Consider the AR(2) process

$$X_{n+1} = \frac{1}{2}X_n + \frac{1}{3}X_{n-1} + \frac{1}{8}\nu_{n+1}$$

Does X_n admit an equilibrium distribution for $n \gg 1$? If so, compute it. [Hint: write the AR(2) equation as a "vector AR(1)" equation introducing the vector variable $Y_n = (X_n, X_{n-1})$. Solve the latter equation iteratively.]

D. Wiener's construction of Brownian motion

Consider the following family of functions

$$X_N(t) = \frac{\nu_0 t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}}\sum_0^N \frac{1}{n}\left(\nu_n(1 - cosnt) + \nu_n' sinnt\right) \tag{1}$$

where ν_k, ν_k' are i.i.d. N(0,1). Show that $X_N(t)$ is a Gaussian process and calculate $E(X_N(t))$, $E(X_N(t)X_N(s))$. Tak the limit as $N \to \infty$ of the covariance and argue, heuristically, that

$$X(t) = \frac{\nu_0 t}{\sqrt{2\pi}} + \frac{1}{\sqrt{\pi}} \sum_0^\infty \frac{1}{n} \left(\nu_n(1 - cosnt) + \nu'_n sinnt\right) \tag{2}$$

represents a Brownian path. This construction was proposed by N. Wiener in 1923.

E. Exponential Martingale. Let $B(t)$ be a Brownian motion. Show that

$$M_\sigma(t) = e^{\sigma B(t) - \frac{\sigma^2 t}{2}} \tag{3}$$

is a martingale. Deduce that

$$X^{(m)}(t) = \left[\frac{d^m}{d\sigma^m}\right]_{\sigma=0} M_\sigma(t) \tag{4}$$

is a martingale for all $m \geq 1$. Compute explicitly $X^{(m)}(t)$ for $m = 1, 2, 3$.