PDE in Finance, Spring 2009: Homework 4
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1. CEV Model.

(a) Create a finite-difference scheme (or trinomial tree) that computes option prices for the CEV model

$$\frac{dS}{S} = \sigma_0 \left(\frac{S}{S_0}\right)^{\beta} dZ + (r-d)dt \tag{1}$$

with $\sigma_0 = 40\%, \beta = -4, S(t=0) = S_0 = 100, r = 0.25\%, d = 3.30\%$.

(b) Compute the implied volatility curves generated by this model, for strikes in the range of ± 25 deltas (European-style options), for maturities $T = 0.25, 0.5, 0.75, 1$. Measure the slopes (γ) of the implied volatility curves near $S = S_0$ and compare them in each case with the "half-slope" estimate $\gamma \approx \frac{\beta}{2}$.

2. Varadhan Approximation for CEV

Use the previous trinomial scheme to define the forward Fokker-Planck equation and compute the probability distribution function corresponding to the CEV model defined for $T = 0.5$. Define precisely the Riemannian metric associated with the CEV model in exercise 1 and compute the Varadhan approximation for the probability density. Compare it with the numerical solution of FFP equation.

3. Stochastic Volatility Model

Consider the stochastic volatility model

$$\begin{aligned} \frac{dS}{S} &= \sigma dZ + (r-d)dt \\ \frac{d\sigma}{\sigma} &= \kappa dW, \quad E(dW dZ) = \rho dt, \end{aligned} \tag{2}$$

where ρ, κ are constants.

(a) Assume $\sigma_0 = 40\%, S_0 = 100, \beta = \frac{\kappa\ \rho}{\sigma_0} = -4, d = 3.30\%, r = 0.25\%$ (similarly to Problem 1.) (a) Using Monte Carlo simulation, compute the implied volatility curves associated with the model at expirations $T = 0.25, 0.5, 0.75, 1.0$.

(b) Compare the curves that you obtained with the ones in Problem 1.

4. Riemannian distance and SV model.

Consider the SV model in Problem 2. Show that the Riemannian distance associated to this model is

$$dL^2 = \frac{1}{1-\rho^2} \cdot \frac{\kappa^2 dx^2 - 2\kappa\rho dx d\sigma + d\sigma^2}{\kappa^2 \sigma^2}.$$

(b) Show that if you make the change of variables

$$z = \frac{\kappa x - \rho\sigma}{\sqrt{1-\rho^2}}$$

the associated distance becomes exactly the Poincare metric

$$dL^2 = \frac{1}{\kappa^2} \cdot \frac{dz^2 + d\sigma^2}{\sigma^2}, \tag{3}$$

(c) Let $\overline{x} > 0$ be a given number. Compute the length, $L^*(\overline{x})$, of the shortest geodesic connecting the point $(0, \sigma_0)$ to the set $\{(x, \sigma) : x \geq \overline{x}\}$. Hint: change variables, as in (b), and show that this is the same problem as computing the shortest distance from the point

$$\left(\frac{-\rho\sigma_0}{\sqrt{1-\rho^2}}, \sigma_0 \right)$$

to the half-space

$$\left\{ (z, \sigma), z\sqrt{1-\rho^2} + \sigma\rho \geq \kappa\overline{x} \right\}.$$

(d) Show that the center of circle corresponding to the optimal geodesic is located at the point $(\frac{\kappa\overline{x}}{\sqrt{1-\rho^2}}, 0)$, and that the length of the geodesic segment between the point and the half-space is

$$L^* = \frac{1}{\kappa} \int_{\theta_i}^{\theta_f} \frac{du}{sinu}, \tag{4}$$

where

$$\theta_i = \arctan\left(\frac{\sigma_0\sqrt{1-\rho^2}}{\kappa\overline{x} + \rho\sigma_0} \right), \quad \theta_f = \arctan\left(\frac{\sqrt{1-\rho^2}}{\rho} \right).$$

(e) Conclude that

$$L^*(\overline{x}) = \frac{1}{\kappa} \left| \ln\left[\frac{\kappa\overline{x} + \rho\sigma_0 + \sqrt{(\kappa\overline{x} + \rho\sigma_0)^2 + \sigma_0^2(1-\rho^2)}}{\sigma_0(1+\rho)} \right] \right|, \tag{5}$$

and derive a formula for the implied volatility skew in the limit $\sigma_0^2 T \ll 1$.

(f) Evaluate the formula for the numerical values of the parameters given above.

5. Taking into account the "wings". Consider the approximation for the implied volatility for log-moneyness $\overline{x}$ and time-to maturity T:

$$\sigma(\overline{x}, T) = \frac{2|\overline{x}|}{L^*(\overline{x}) + \sqrt{L^*(\overline{x})^2 + 8T\overline{x}}} \tag{6}$$

Fit the data on SPY options in the accompanying spreadsheet to this formula and deduce the values of the parameters κ, σ_0 and ρ.