PDE in Finance, Spring 2009
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Homework 1

1. Fourier Transform Calculate the Fourier transform of

$$f(x) = \frac{1}{1+x^2}, \tag{1}$$

$$f(x) = sin(2\pi x), \tag{2}$$

and

$$f(x) = g(x - x_0), \text{where } g \text{ is a given function} \tag{3}$$

2. Well-posedness of the Cauchy problem

(i) Consider a PDE of the form

$$\frac{\partial u}{\partial t} = \sum_{k=0}^{n} a_k \frac{\partial^k u}{\partial x^k}, \quad t > 0, \ x \in (-\infty, +\infty) \tag{4}$$

where $a_0, a_1, ..., a_n$ are constants. Using Fourier Transform, give a "formal" solution of the Cauchy problem with initial condition $u(x, 0) = u_0(x)$.

(ii) As in the above, consider the case

$$\frac{\partial u}{\partial t} = -\frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2}, \quad u(x, 0) = u_0(x). \tag{5}$$

Argue that, despite the existence of a "formal" solution, the Cauchy problem in (5) is not well-posed for the equation above, i.e., show that a solution for a typical initial condition will blow-up or depend sensitively on the initial conditions. [Hint: Fourier. Also, try solving equation (5) for a Heaviside-type initial condition $u_0(x) = max(x/|x|, 0)$.] What goes wrong ?

(iii) Consider instead the final-value problem for the PDE in (5), i.e. a *terminal* condition

$$u(x,t)|_{t=T} = u_0(x)$$

and the problem defined for $t < T$. In this case, everything works fine. Why?

3. Analytic solution Solve the Cauchy problem for the heat diffusion equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \quad &+ \quad \mu\frac{\partial u}{\partial x} - ru = 0, \quad x \in (-\infty, +\infty), t < T \\ u(x, t = T) \quad &= \quad max(x/|x|, 0) \end{aligned} \tag{6}$$

Assuming $\sigma = 0.2$, $\mu = 0.01$, $r = 0.02$ and $T = 0.5$, compute the value of $u(0.1, 0)$.

Finite-difference scheme. Build a lattice, or trinomial tree, associated with the Cauchy problem (6). Choose δt and h, and calculate the lattice probabilities

$$\begin{aligned} p_u &= Pr.\{X_{n+1} = x + h | X_n = x\} \\ p_d &= Pr.\{X_{n+1} = x - h | X_n = x\} \\ p_m = 1 - p_u - p_d &= Pr.\{X_{n+1} = x | X_n = x\} \end{aligned} \tag{7}$$

associated with the numerical values for σ, μ, r, T of the previous problem. Solve the Cauchy problem using the corresponding discrete backward Fokker-Planck equation and compare the solution to the one of the previous problem.