A Dynamic Model for Hard-to-Borrow Stocks

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March 10, 2009

Abstract

We study the price-evolution of stocks that are subject to restrictions on short-selling, generically referred to as *hard-to-borrow*. Such stocks are either subject to regulatory short-selling restrictions or have insufficient float available for lending. Traders with short positions risk being "bought-in", in the sense that their positions may be closed out by the clearing firm at market prices. The model we present consists of a coupled system of stochastic differential equations describing the stock price and the "buy-in rate", an additional factor absent in standard models. The conclusion of the model is that short-sale restrictions result in increased prices and volatilities. Our model prices options as if the stock paid a continuous dividend, reflecting a modified form of Put-Call parity. Another consequence is that stocks that do not pay a dividend may have calls subject to early exercise. Both features are in agreement with empirical (market) observations on hard-to-borrow stocks.

1 Introduction

Short-selling, the sale of a security not held in inventory, is achieved as follows: (a) the seller indicates to a broker that he wishes to sell a stock that he does not own; (b) the broker arranges for a buyer; (c) the trade takes place. After that, the clearing firm representing the seller must deliver the stock within a stipulated amount of time. To make delivery, the seller must buy the stock in the market or borrow it from a stock-loan desk. *Naked* short-selling means that the sale took place in advance of locating a lender; "regular" short-selling implies that a lender has been found before the trade took place.

The availability of stocks for borrowing depends on market conditions. While many stocks are easily borrowed, others are in short supply. In the latter case, establishing a short position may be costly. In general, hard-to-borrow (HTB) stocks earn a reduced interest rate on cash credited for short positions by the clearing firms. Moreover, short positions in HTBs may be forcibly repurchased (bought in) by the clearing firms. In general, these buy-ins will be made in order to cover shortfalls in delivery of stock following the Securities and Exchange Commission's Regulation SHO[1].

The *short interest* in a stock is the percentage of the float currently held short in the market. Although a stock may have a large short interest without actually being subject to buy-ins, hard-to-borrow stocks are those for which buy-ins will occur with non-zero probability. A trader subject to a potential buy-in is notified by his clearing firm during the trading day. However, he usually remains uncertain of how much, if any, of his short position might be repurchased until the market closes. Typically, buy-ins by clearing firms take place in the last hour of trading, i.e. between 3 and 4 PM Eastern Time. An option trader who has been bought-in will have to sell any unexpected long deltas acquired through buy-ins. As a consequence, someone who is long a put will not have the same synthetic position as the holder of a call and short stock. The latter position will reflect an uncertain amount of short stock overnight but not the former.

While buy-ins take place, it is reasonable to expect that the stock price will be trending upwards. One reason for this is that knowledge of potential buy-ins can lead speculators to run up (buy) the stock. However, once the buy-ins have finished, there is no reason for the stock price to remain elevated. As a general rule, the price will drop after buy-ins are completed.

In many emerging markets, stocks may be impossible to short due to local regulations. Even in developed markets with liberal short-selling rules, a situation may arise in which lenders can demand physical possession of the stock. In this case, the stock price may appear to be "pumped up" by forced buying of short positions in the market. Recent events in 2008 have led to restrictions on naked shorting and bans on regular shorting for many financial stocks. Such restrictions are known to lead to "overpricing", in the sense of Jones and Lamont (2002).

Some key elements of the world of hard-to-borrows can be readily identified. The larger the short interest, the harder it is to borrow stock. Another consideration is that shorting stock and buying puts are not equivalent as a means of gaining short exposure. This last point is critical for understanding, valuing and trading HTBs.

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[1]The Wikipedia entry for Reg SHO states: "The SEC enacted Regulation SHO in January 2005 to target abusive naked short selling by reducing failure to deliver securities, and by limiting the time in which a broker can permit failures to deliver. In addressing the first, it stated that a broker or dealer may not accept a short sale order without having first borrowed or identified the stock being sold. The rule had the following exemptions:

(i) Broker or dealer accepting a short sale order from another registered broker or dealer, (ii) Bona-fide market making, (iii) Broker-dealer effecting a sale on behalf of a customer that is deemed to own the security pursuant to Rule 200 through no fault of the customer or the broker-dealer." For more information and updates on Reg SHO, the reader should consult the Securities and Exchange Commission website www.sec.gov.

The following examples illustrate the rich variety of phenomena associated with HTBs, which we will attempt to explain with our model.[2]

1. Hard-to-borrowness and the cost of conversions. In January 2008, prior to announcing earnings, the stock of VMWare Corp. (VMW) became extremely hard-to-borrow. This was reflected by the unusual cost of converting on the Jan 2009 at-the-money strike. *Converting* means selling a call option and buying a put option of the same strike and 100 shares of stock. According to Put-Call Parity, for an ordinary (non-dividend paying) stock, the premium-over-parity of a call (C_{pop}) should exceed the premium-over-parity of the corresponding put (P_{pop}) by an amount approximately equal to the strike times the spot rate[3]. In particular, a converter should receive a credit for selling the call, buying the put and buying 100 shares. However, for hard-to-borrow stocks the reverse is often true. For VMW, the difference $C_{pop} - P_{pop}$ for the January 2009 \$60 line was a whopping -\$8.00! A converter would therefore need to *pay* \$8 (per share) to enter the position, i.e. \$800 per contract.

Following the earnings announcement, VMW fell roughly \$28. At the same time, the cost of the conversion on the 60 strike in Jan 2009 dropped in absolute value to approximately -\$1.80 (per share) from -\$8.00. (The stock was still HTB, but much less so.) Therefore, a trader holding 10 puts, long 1000 shares and short 10 calls, believing himself to be delta-neutral, would have lost (\$8.00-\$1.80)$\times 10 \times 100 =$ \$6200.

2. Artificially high prices and sharp drops. Over a period of less than two years, from 2003-2005, the stock of Krispy Kreme Donuts (KKD) made extraordinary moves, rising from single digits to more than \$200 after adjusting for splits[4]. During this time, buy-ins were quite frequent. Short holders of the stock were unpredictably forced to cover part of their shorts by their clearing firms, often at unfavorable prices. Subsequent events led to the perception by the market that accounting methods at the company were questionable. After 2005, Krispy Kreme Donuts failed to report earnings for more than four consecutive quarters and faced possible delisting. At that time, several members of the original management team left or were replaced and the stock price dropped to less than \$3. In a companion paper, we will argue that HTB stocks have erratic prices which often rise fast and are subject to "crashes".

United Airlines filed for Chapter 11 protection at the end of 2002 with debts far exceeding their assets. Nevertheless, the stock price for United continued to trade above \$1 with extremely frequent buy-ins for more than 2 years.

3. Unusual pricing of vertical spreads[5]. Options on the same HTB

[2] These examples are provided from the second authors' personal experience trading options. Most of the prices can be recovered from publicly available data sources.

[3] Premium-over-parity(POP) means the difference between the (mid-)market price of the option and its intrinsic value. Some authors also call the POP the *extrinsic value*. We use "approsimately equal" because listed options are American-style, so they have an early exercise premium. Nevertheless, at-the-money options will generally satisfy the Put-Call Parity equation within narrow bounds.

[4] There were two 2:1 splits for this stock in its lifetime and both took place between 2003 and 2005.

[5] A vertical spread (see Natenberg(1998)) is defined as a buying an option with one strike

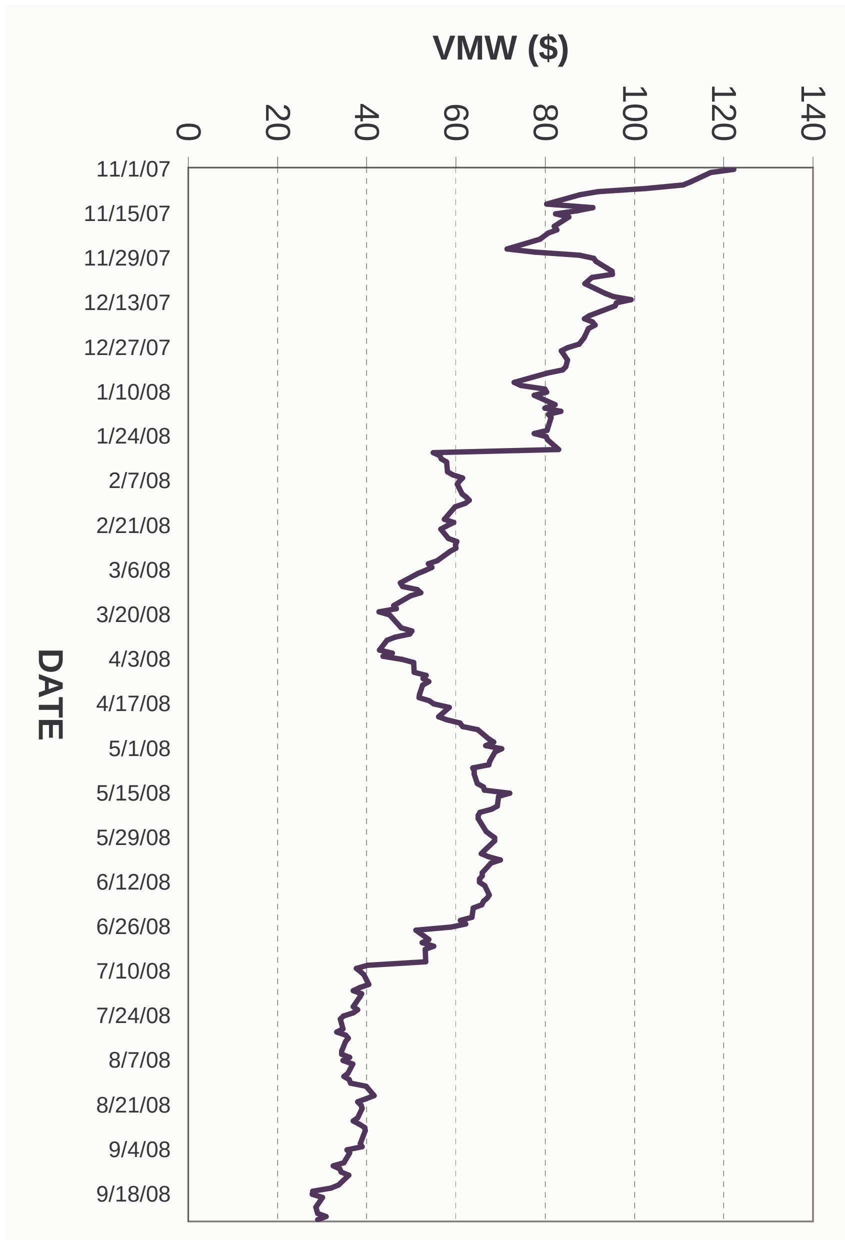

Figure 1: Closing prices of VMWare (VMW) from November 1, 2007 until September 26, 2008. The large drop in price after earnings announcement in late January 2008 was accompanied by a reduction in the difficulty to borrow, as seen in the price of conversions.

name with different strikes and the same expiration seem to be mispriced. For example the biotech company Dendreon (DNDN) was extremely hard-to-borrow in February 2008. With stock trading at \$5.90, the January 2009 2.50-5.00 put spread was trading at \$2.08 (midpoint prices), shy of a maximal value of \$2.50, despite having zero intrinsic value. Notice this greatly exceeds the "midpoint-rule" value of \$1.25 which is typically a good upper bound for out-of-the-money verticals.

4. Short-squeezes. A short-squeeze is often defined as a situation in which an imbalance between supply and demand causes the stock to rise abruptly and a scramble to cover on the part of short-sellers. The need to cover short positions drives the stock even higher. In a recent market development Porsche AG indicated its desire to control 75% of Wolkswagen, leading to an extraordinary spike in the stock price (see Figure 2).

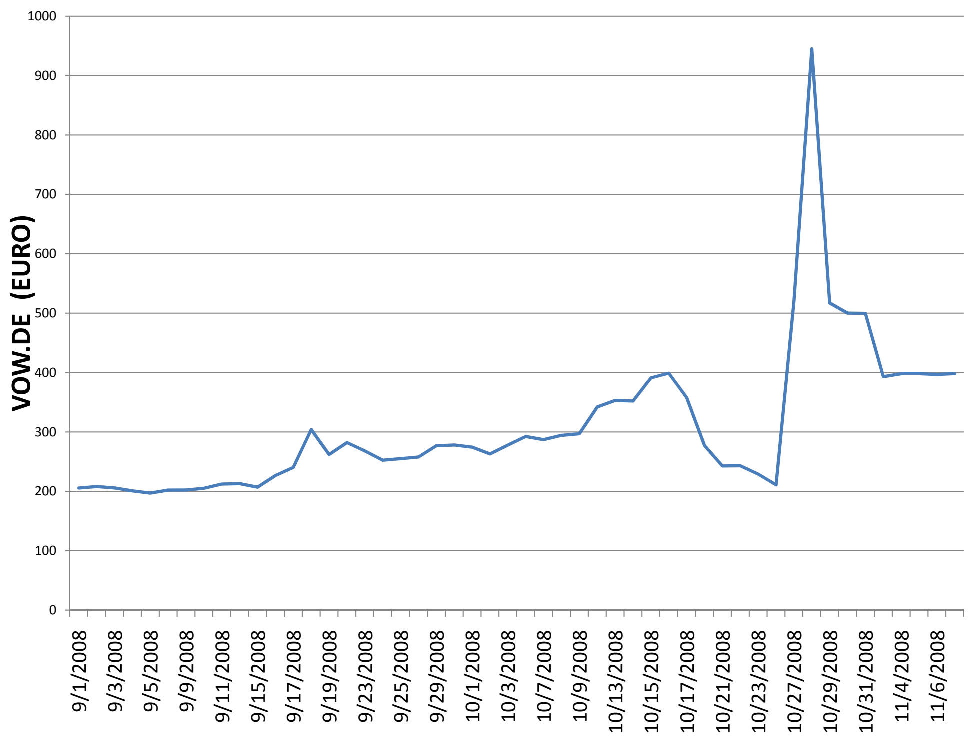

Figure 2: Short-squeeze in Volkswagen AG, October 2008.

and selling another with a different strike on the same series.

To recover these features within a mathematical model, we propose a feedback mechanism that couples the dynamics of the stock price with the frequency at which buy-ins take place, *viz.* the *buy-in rate*. The buy-in rate represents the frequency of buy-in events to which the stock is subjected, measured in events/year. Thus, a buy-in rate of 52 corresponds to a stock that is subjected to a buy-in once per week. In our model, buy-ins are stochastic, so the frequency does not indicate a regular pattern, but rather an expected number of buy-in events per year.

When a buy-in takes place, firms repurchase stock in the amount of the undelivered short positions of their clients. This introduces an excess demand for stock that is unmatched by supply at the current price, resulting in a temporary upward impact on prices.[6] Each day, when buy-ins are completed, the excess demand disappears, causing the stock price to jump roughly to where it was before the buy-in started. (See Figures 2 and 3). We model the excess demand as a drift proportional to the buy-in rate and the relaxation as a Poisson jump with intensity equal to the buy-in rate, so that on average, the expected return from holding stock which is attributable to buy-in events is zero.[7]

Although a link may exist between the short interest and the buy-in rate, we avoid, at the modeling level, having to produce a definite form for this relation. We note that they should vary in the same direction: the greater the short interest, the more frequent the buy-ins. The more frequent the buy-ins, the higher the stock price gets driven by market impact. Accordingly, the feedback alluded to above is modeled by coupling directly the buy-in rate variations to changes in the stock price.

The model presented here adds to a considerable amount of previous work on hard-to-borrows. On the theoretical side, we mention Nielsen (1989), Duffie et al. (2002) and Diamond and Verrecchia (1987). On the empirical side, we mention Lamont and Thaler (2003), Jones and Lamont (2002), Lamont (2004), Charoenrook and Daouk (2005) and Evans *et al* (2008).

The novelty in our approach vis-a-vis the papers mentioned above is that we introduce a new stochastic process to describe the asset price based on the (variable) intensity of buy-ins. Using this process, we can derive option pricing formulas and describe many stylized facts. This is particularly relevant to the study of how options markets and short-selling interact. The recent article by Evans et al (2008) covers empirical aspects of the problem of short-selling HTB stocks from the point of view of option market-makers, which is also one of the considerations of our model, via the buy-in rate. Our model can be seen as providing a dynamic framework for quantifying losses for market-makers due to buy-ins alluded to in Evans *et al.*

[6]Professionals who were bought-in may need to re-establish their shorts (for example to hedge options). Furthermore, an increase in price may attract additional sellers at the new higher price, potentially increasing the short interest and the buy-in activity.

[7]This assumption states mathematically that the stock has zero expected return in the "physical", or "subjective" measure. For option pricing, cost-of-carry considerations apply and the probability distribution is modified accordingly, as explained in Section 3. The results would not be affected if we assumed instead a non-zero drift for the stock price.

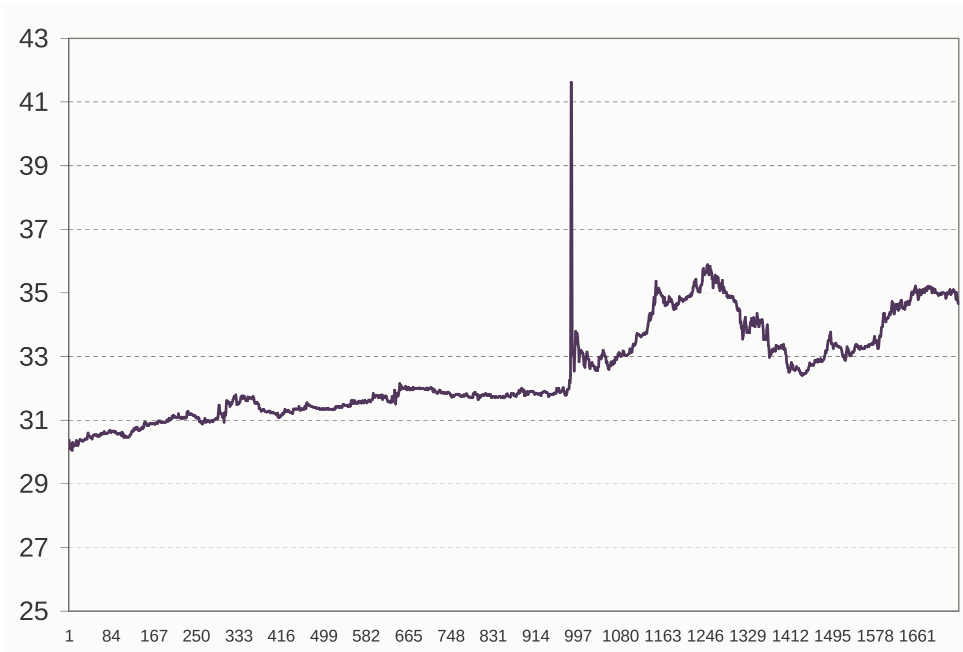

Figure 3: Minute-by-minute price evolution of Interoil Corp. (IOC) between June 17 and June 23, 2008. Notice the huge spike, which occurred on the closing print of June 19th. The price retreats nearly to the same level as prior to the buy-in.

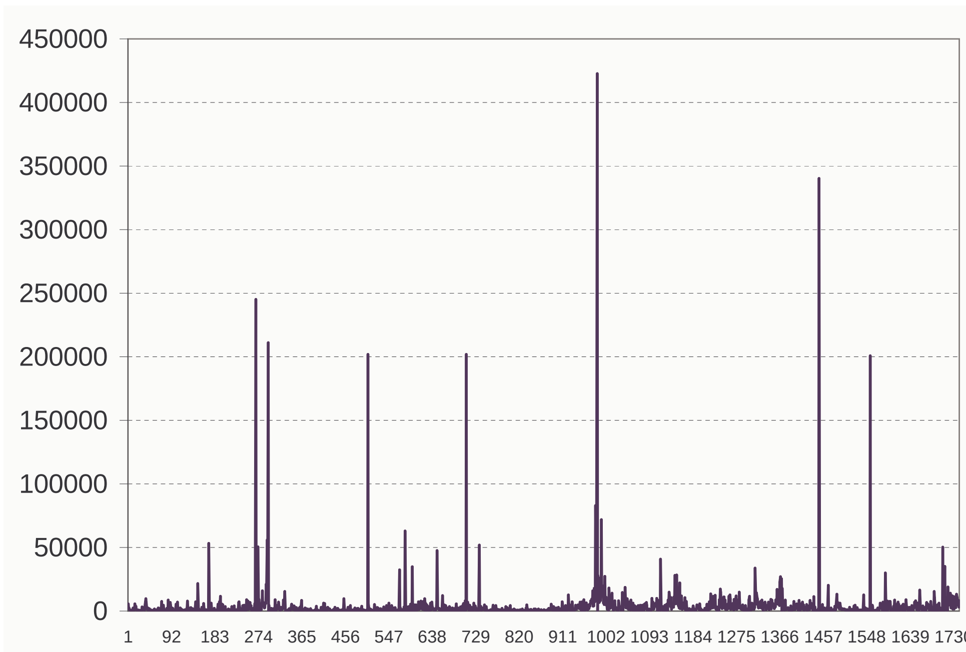

Figure 4: Minute-by-minute share volume for IOC between June 17 and June 23, 2008. The average daily volume is approximately 1.3 million shares; the volume on the last print of 6/19 was 422,600 shares. The final print was entirely due to the buy-in.

The rest of the paper is organized as follows: in Section 2, we give mathematical form to the model. In Section 3 we show how this model leads to a risk-neutral measure for pricing options. In the pricing measure, the effect of buy-ins is seen as a *stochastic dividend yield*, which reflects that holders of long stock can, in principle "lend it" for a fee to traders that wish to maintain short positions and not risk buy-ins. Using this model we can derive mathematical formulas for forward prices, and a corresponding Put-Call Parity relation which is consistent with the new forward prices and matches the observed conversion prices. We emphasize that even though Put-Call Parity does not hold with the nominal rate of interest and dividend, it holds under the new pricing measure, so there exists an equilibrium pricing of options. The anomalous vertical spreads are thereby explained as well. In Section 4 we present an option pricing formula for European options and tractable approximations for Americans. One of the most striking consequences of the study is the early exercise of deep in-the-money calls. In Section 5, we observe that the fluctuations in the intensity of buy-ins and changes in hard-to-borrowness can be measured using leveraged ETF tracking financial stocks (which were extremely hard to borrow in the fall of 2008). Conclusions are presented in Section 6.

2 The model

We present a model for the evolution of prices of hard-to-borrow stocks in which S_t and λ_t denote respectively the price and the buy-in rate at time t. Recall that the buy-in rate is assumed to be proportional to (or to vary in the same sense as) the short-interest in the stock.

We assume that S_t and λ_t satisfy the system of coupled equations

$$\frac{dS_t}{S_t} = \sigma\, dW_t + \gamma\lambda_t dt - \gamma\, dN_{\lambda_t}(t) \qquad (1)$$

$$dX_t = \kappa dZ_t + \alpha\left(\overline{X} - X_t\right) dt + \beta\frac{dS_t}{S_t}\ ,\ X_t = ln\left(\lambda_t/\lambda_0\right), \qquad (2)$$

where $dN_\lambda(t)$ denotes the increment of a standard Poisson process with intensity λ over the interval $(t, t+dt)$.[8] The parameters σ and γ are respectively the volatility and the price elasticity of demand due to buy-ins; W_t is a standard Brownian motion. Equation (2) describes the evolution of the logarithm of the buy-in rate; κ is the volatility of the rate, Z_t is a Brownian motion, $\overline{X}$ is a long-term equilibrium value for X_t, α is the speed of mean-reversion and β couples the change in price with the buy-in rate. We assume that Z_t, the Brownian driving the buy-in rate fluctuations, is independent of W_t, which drives the stock price.[9]

[8] Poisson increments corresponding to different time-intervals are independent and we have Prob. $\{dN_{\lambda_t}(t) = 1\} = \lambda_t\, dt + o(dt)$; Prob. $\{dN_{\lambda_t}(t) = 0\} = 1 - \lambda_t\, dt + o(dt)$.

[9] The independence of W_t and Z_t is immaterial: the interesting coupling between buy-in rate and price occurs via the parameter β.

We assume that $\beta > 0$; in particular $x = ln(\lambda)$ is positively correlated with price changes. This is the key feature of our model because it introduces a positive feed-back between increases in buy-ins (hence in short-interest in the stock) and price increases.

Equations (1) and (2) describe the evolution of the stock price across an extended period of time. One can think of a diffusion process for the stock price, which is punctuated by jumps occurring at the end of the trading day, the magnitude and frequency of the latter being determined by λ. Fluctuations in λ_t represent the fact that a stock may be difficult to borrow one day and easier another. In this way, the model describes the dynamics of the stock price as costs for stock-loan vary. Short squeezes can be seen as events associated with large values of λ, which invariably will exhibit price spikes (rallies followed by a steep drop).

An examination of Figures 3 and 4 show that a buy-in event often looks like an upward jump followed immediately by a retracing downward jump. This might lead one to propose a model with coupled jumps of both signs.[10] In the HTB context, the losses due to buy-ins are associated with downward price jumps: hence, a model consisting only of downward jumps already captures this effect.

3 The cost of shorting: buy-ins and effective dividend yield

The Securities and Exchange Commission's Regulation SHO for threshold securities requires that traders "locate" shares that they intend to short before doing so. Thus, if a trader wishes to sell short 10,000 shares of VMWare, he must ask his clearing firm to borrow 10,000 shares, either among the firm's inventory or through a stock-loan transaction. Firms usually charge a fee, usually in the form of a reduced interest rate, to accommodate clients who wish to short hard-to-borrows. In practice, this "rate" is often negative, so there is a cost associated with maintaining a short position.

Option market-makers need to hedge by trading the underlying stock, both on the long and short side, with frequent adjustments. However, securities that become hard to borrow are subject to buy-ins as the firm needs to deliver shares according to the presently existing settlement rules. Form a market-maker's viewpoint, a hard-to-borrow stock is essentially a security that presents an increased likelihood of buy-ins.

The profit or loss for a market-maker is affected by whether and when his short stock is bought in and at what price. Generally, this information is not known until the end of the trading day. To model the economic effect of buy-ins, we assume that the trader's PNL from a short position of one share over a period $(t, t + dt)$ is

[10] Such "spike" models are often used in the electricity derivatives literature; see Blanco and Soronow (2001), Borovkova and Permana (2006), De Jong and Huisman (2002).

$$\text{PNL} = -dS_t - \xi\,\gamma\,S_t = -S_t\,(\sigma\,dW_t \ + \lambda_t\gamma\,dt)\,,$$

where $\text{Prob.}\{\xi = 0\} = 1 - \lambda_t\,dt + o(dt)$ and $\text{Prob.}\{\xi = 1\} = \lambda_t\,dt + o(dt)$. Thus, we assume that *the trader who is short the stock does not benefit from the downward jump in equation (1)* because he is no longer short by the time the buy-in is completed. The idea is that the short trader takes an economic loss post-jump due to the fact that his position was closed at the buy-in price.

Suppose then, hypothetically, that the trader was presented with the possibility of "renting" the stock for the period $(t, t+dt)$ so that he or she can remain short and be guaranteed not to be bought in. The corresponding profit and loss would now include the negative of the downward jump *i.e.* $\gamma\,S_t$ if the jump happened right after time t. Since jumps and buy-ins occur with frequency λ_t, the expected economic gain is $\lambda_t\,\gamma\,S_t$. It follows that the fair value of the proposed rent is $\lambda_t\,\gamma$ per dollar of equity shorted. In other words, $\lambda_t\,\gamma$ can be viewed as the cost-of-carry for borrowing the stock.

Since shorts pay rent, longs collect it. Hence, we can interpret $\lambda_t\,\gamma$, as a convenience yield associated with owning the stock when the buy-in rate is λ_t. Traders who are long the stock can lend it to traders willing to pay a fee to maintain short positions. This convenience yield is monetized by longs lending their stock out for one day at a time and charging the fee associated with the observed buy-in rate (we assume that this fee is observable, for simplicity, and that traders are allowed to enter into such stock-lending agreements, in the interest of establishing the concept of fair value for shorting within our model).

For options pricing, the convenience yield or rent is mathematically equivalent to a stochastic dividend yield which is credited to long positions and debited from shorts who enter into lending agreements. For traders who are short but do not enter into such agreements, it is assumed that stochastic buy-ins prevent them from gaining from downward jumps.

Notice that, statistically, the economic costs of paying rent or risking buy-ins are equivalent. In particular, the cost of carrying (or financing) stock can be quantified in terms of λ and the interest rate. We can therefore introduce an arbitrage-free pricing measure associated with the physical process (1)-(2), which takes into account the rent, or stock-financing. Based on the fundamental model for the dynamics of prices, this equation should take the form:

$$\frac{dS_t}{S_t} = \sigma dW_t + r\,dt - \gamma dN_{\lambda_t}(t), \tag{3}$$

where r is the instantaneous interest rate. The absence of the drift term $\lambda_t\gamma$ in this last equation is due to the fact that, under an arbitrage pricing measure, the price process adjusted for dividends and interest is a martingale.

Notice that the rent $\lambda\gamma$ cancels exactly the drift component of the model and gives rise to the risk-neutral model in which the expected return is equal to the cost of carry. [11]

[11] Clearly, the assumption that the shorts don't collect the jump, which has magnitude γ, results in the fact that the rent $\lambda\gamma$ is exactly equal to the drift in (1). We could have assumed

The first application of the model concerns forward pricing. Assuming constant interest rates, we have

$$\begin{aligned}
\text{Forward Price} &= E\{S_T\} \\
&= E\left\{ S_0\, e^{\sigma W_T - \frac{\sigma^2 T}{2} + rT}\ (1-\gamma)^{\int_0^T dN_{\lambda_t}(t)} \right\} \\
&= S_0 e^{rT} E\left\{ e^{-\int_0^T \lambda_t dt} \sum_k \frac{(\int_0^T \lambda_t dt)^k}{k!} (1-\gamma)^k \right\} \\
&= S_0\, e^{rT} E\left\{ e^{-\gamma \int_0^T \lambda_t dt} \right\}. \qquad (4)
\end{aligned}$$

This equation gives a mathematical formula for the forward price in terms of the buy-in rate and the scale constant γ. Clearly, if there are no jumps, the formula becomes classical. Otherwise, notice that the dividend is positive and delivering stock into a forward contract requires hedging with less than one unit of stock, "renting it" along the way to arrive at one share at delivery. From equation (4), the term-structure of forward dividend yields (d_t) associated to the model is given by

$$e^{-\int_0^T d_t dt} = E\left\{ e^{-\gamma \int_0^T \lambda_t dt} \right\}. \qquad (5)$$

4 Option Pricing for Hard-to-Borrow Stocks

Put-Call Parity for European-style options states that

$$C(K,T) - P(K,T) = S(1-DT) - K(1-RT),$$

where $P(K,T), C(K,T)$ represent respectively the fair values of a put and a call with strike K and maturity T, S is the spot price and R, D are respectively the simply discounted interest rate and dividend rate.[12] It is equivalent to

$$C_{pop}(K,T) - P_{pop}(K,T) = KRT - DST \qquad (6)$$

instead that the expected loss of revenue from buy-ins is $\omega\lambda_t\, S_t$, where ω is another constant of proportionality. Although this more general assumption changes the mathematics slightly, the practical implications – existence of an effective dividend yield – are the same.

[12] In this section we use the Put-Call Parity formula used by traders, with simply discounted rates, since this is the market convention for equity derivatives. The same applies to the version of Put-Call parity in terms of premium-over-parity.

where $P_{pop}(K,T) = P(K,T) - max(K-S,0)$ represents the premium over parity for the put, a similar notation applying to calls.

It is well-known that Put-Call parity does not hold for hard-to-borrow stocks if we enter the nominal rates and dividend rates in equation (6). The price of conversions in actual markets should therefore reflect this. Whereas a long put position is mathematically equivalent to being long a call and short 100 shares of common stock, this will not hold if the stock is a hard-to-borrow. The reason is that shorting costs money and the arbitrage between puts and calls on the same line, known as a *conversion*, cannot be made unless there is stock available to short. Conversions that look attractive, in the sense that

$$C_{pop}(K,T) - P_{pop}(K,T) < KRT - DST, \tag{7}$$

may not result in a riskless profit due to the fact that the crucial stock hedge (short 100 shares) may be impossible to establish.

We quantify deviations from Put-Call Parity by considering the function

$$d_{imp}(K,T) \equiv \frac{C_{pop}(K,T) - P_{pop}(K,T) - KRT}{-ST}, \quad 0 < K < \infty. \tag{8}$$

As a function of K, $d_{imp}(K,T)$ will be approximately flat for low strikes and will rise slightly for large values of K because puts become more likely to be exercised.[13] The dividend yield for the stock should correspond roughly to the level of $d_{imp}(K,T)$ for at-the-money strikes. If we consider American options on dividend-paying stocks or exchange-traded funds (e.g. SPY), then the implied dividend curve will, in addition, be lower for low strikes, reflecting the fact that calls have an early-exercise premium.

The situation is quite different for hard-to-borrow stocks as we can see from Figures 5 and 6. Two distinctions are important: (i) the implied dividend curve $d_{imp}(K,T)$ for $K \approx S$ is not equal to the nominal dividend yield (which is zero, in the case of the stocks that are displayed in the figures). Instead, it has a positive value. (ii) The implied dividend curve $d_{imp}(K,T)$ also bends for low values of the strike, suggesting that calls with low strikes should have an early exercise premium.

The first feature – a change in level in the implied dividend curve – has to do with the extra premium for being long puts in a world where shorting stock is difficult or expensive. Since synthetic puts cannot be manufactured by owning calls and shorting stock, the nominal put-call parity does not hold. Instead, is is replaced by a *functional* put-call parity, which expresses the relative value of puts and calls via an effective dividend rate. Indeed, if we define

$$D^*(T) = d_{imp}(S,T),$$

i.e. the at-the-money implied dividend yield, we obtain, using the definition of d_{imp}, the new parity relation

[13] Of course, if the options are European-style, then $d_{imp}(K,T) = D$, the dividend yield. Unfortunately for the theorists among us, listed equity options in the U.S. and most of Europe are American-style.

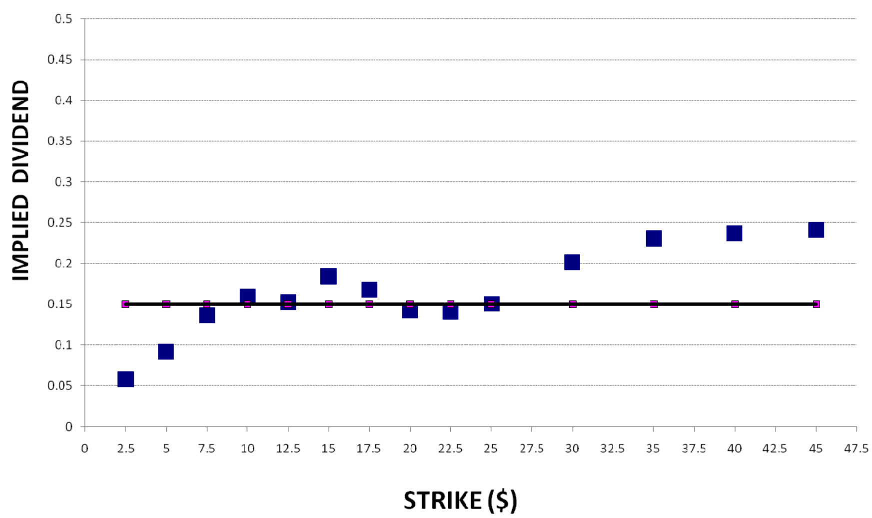

Figure 5: Implied dividend rates as a function of strike price for options on Dendreon (DNDN). The trade date is January 10, 2008 and the expiration is January 17, 2009. The stock price is $5.81. The best fit constant dividend rate is approximately 15%.(Dendreon does not pay dividends.)

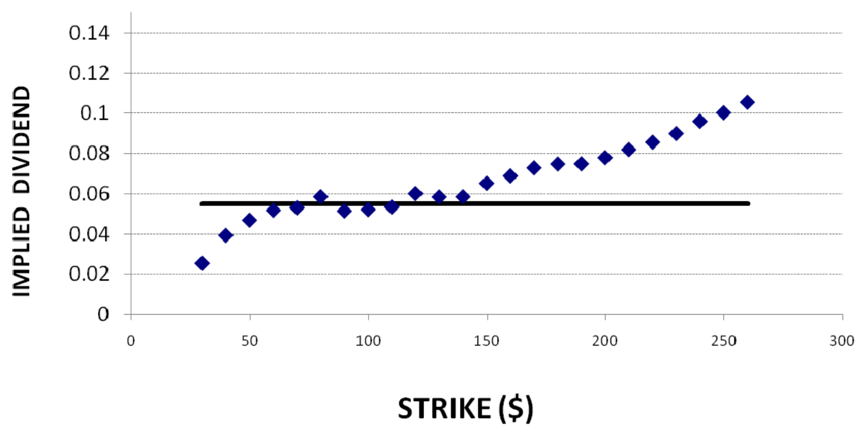

Figure 6: Implied dividend rates for VMWare (VMW). The dates are as in the previous figure and the stock price is $ 80.30. The best fit dividend rate (associated with ATM options) is 5.5%. (VMWare does not pay dividends.)

$$C_{pop}(K,T) - P_{pop}(K,T) = KRT - D^*(T)ST.$$

Here $D^*(T)$ is essentially an implied dividend rate obtained from the options market.

According to our model, we have, from equation (8),

$$D*(T) = \frac{1 - e^{-\int_0^T d_t dt}}{T} = \frac{1}{T} E\left\{ e^{-\gamma \int_0^T \lambda_t dt} \right\}, \tag{9}$$

which connects the implied dividend rate obtained from the options markets to the buy-in rate process.

Empirically, the option market predicts different borrowing rates over time for any given stock. There are several ways to see this. First, through variations in the interest rate (short rate) quoted by clearing firms, and second by conversion-reversals quoted by option market-makers.[14] The latter approach suggests different implied dividends per option series, i.e. contains market expectations of the varying degree of difficulty of borrowing a stock in the future. We can use the model (1)-(2) and equation (9) to calculate a term-structure of effective dividends (or, equivalently, short rates) which could be calibrated to any given stock. To generate such a term-structure, we simulate paths of $\lambda_t,\ 0 < t < T_{max}$ and calculate the discount factors by Monte-Carlo. Figure 8 shows a declining term-structure, which is typical of most stocks. This decay represents the fact that stocks rarely remain HTB over extremely long time periods.

We now derive option-pricing formulas. It follows from Equation (3) that the stock price in the risk-neutral world can be written as

$$S_t = S_0\ M_t\ (1-\gamma)^{\int_0^t dN_{\lambda_t}(t)}, \tag{10}$$

where

$$M_t \ = \ exp\left\{\sigma W_t - \frac{\sigma^2 t}{2} + rt\right\}$$

is the classical lognormal process such that $e^{-rt}M_t$ is a martingale. The third factor in equation (10) represents the effects of buy-ins. If we make the approximation that λ_t is independent of W_t, in the sense that

$$d(log\lambda_t) = \kappa dZ_t + \alpha\left(\overline{log(\lambda)} - log(\lambda_t)\right)dt + \beta\left(\gamma\lambda_t dt - \gamma dN_{\lambda_t}(t)\right),$$

the model becomes more tractable. In this case, we obtain semi-explicit pricing formulas for European-style puts and calls as series expansions by separation of variables. To see this, we define the weights

[14]The short rate, or rate applied to short stock positions, can be viewed as the difference between the riskless rate (Fed funds) and the effective dividend.

$$\begin{aligned}\Pi(n,T) &= \text{Prob.}\left\{\int_0^T dN_{\lambda_t}(t)=n\right\}\\ &= E\left\{e^{-\int_0^T \lambda_t dt}\frac{\left(\int_0^T \lambda_t dt\right)^n}{n!}\right\}\end{aligned} \tag{11}$$

(Figure 7 shows the weights for a particular set of parameter values.)

Denote by $BSCall(s,t,k,r,d,\sigma)$ the Black-Scholes value of a call option for a stock with price s, time to maturity t, strike price k, interest rate r, dividend yield d and volatility σ. We then have

$$C(S,K,T)=\sum_0^\infty \Pi(n,T)BSCall\left(S(1-\gamma)^n,T,K,r,0,\sigma\right), \tag{12}$$

with a similar formula holding for European puts.

Notice that equation (10) can be viewed as the risk-neutral process for a stock that pays a discrete dividend γS_t with frequency λ_t. Therefore, calls will be exercisable if they are deep enough in-the-money. A heuristic explanation is that a trader long a call and short stock would suffer repeated buy-ins costing more than the synthetic put forfeited by exercising. Unfortunately, pricing an American call using the full model (3) entails a high-dimensional numerical calculation, because the number of jumps until time t, $\int_0^t dN_{\lambda_t}(t)$, is not a Markov process unless λ_t is constant. In other words, the state of the system depends on the current value of λ_t and not just on the number of jumps that occurred previously. The case λ_t =constant is an exception; it corresponds to $\beta=0$, i.e. to the absence of coupling between the price process and the buy-in rate. The calculation of American option prices in this case is classical; see for instance Amin (1993). Figure 8 shows the curve $d_{imp}(K,T)$ for American options using the model, consistent with the observed graphs of DNDN and VMW (Figures 4 and 5).

5 Observing hard-to-borrowness in leveraged short ETFs

In this section we show how hard-to-borrowness can also be observed, in some cases, from price data for leveraged long and short ETFs.[15] Since short ETFs maintain a short position in the underlying security, we expect that the cost of

[15] To our knowledge, leveraged ETF were first introduced in early 2007 by ProShares. Direxion, another ETF issuer introduced triple leveraged ETFs since November 2008.

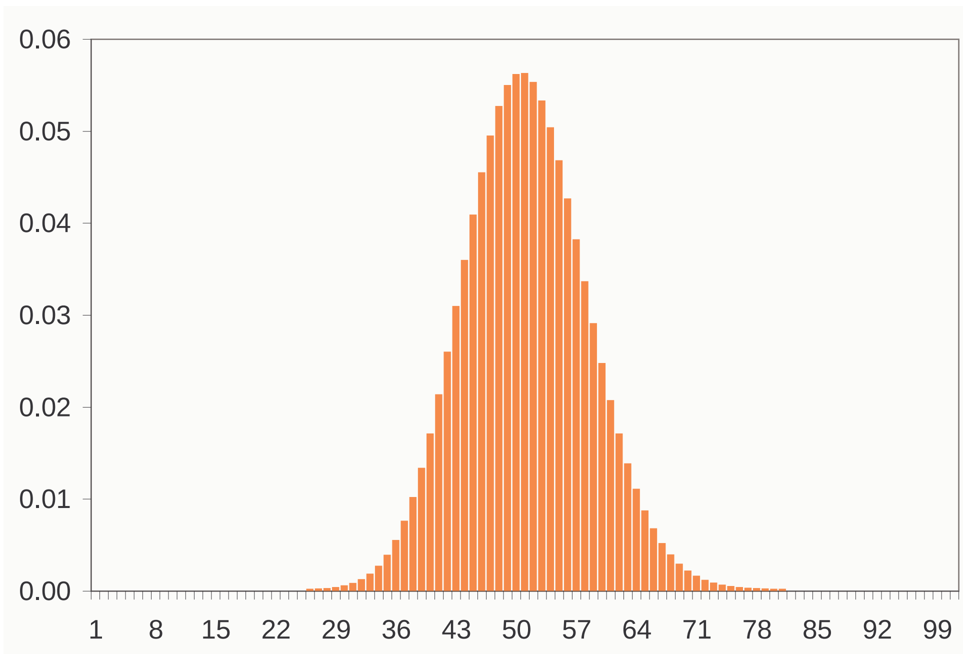

Figure 7: Weights $\Pi(n, T)$ computed by Monte Carlo simulation. The parameter values are $\beta = 1.00, \lambda_0 = 50, T = 0.5yrs., \gamma = 0.03$.

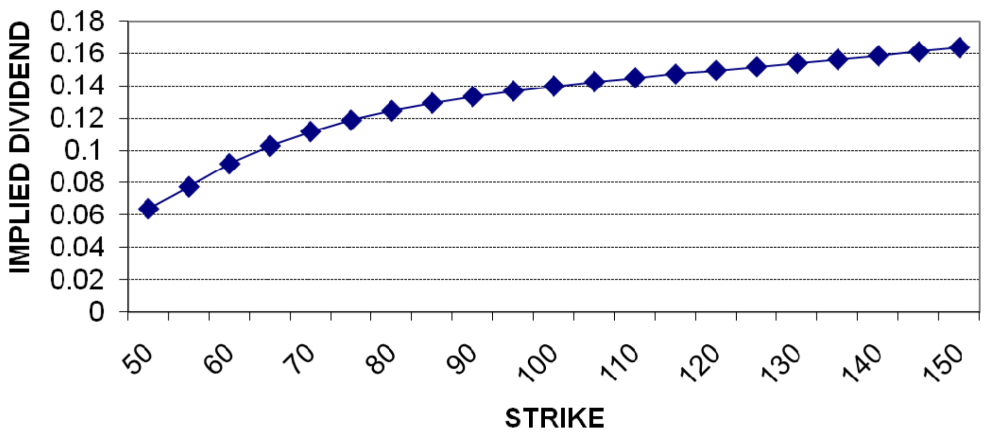

Figure 8: Theoretical implied dividend yield $d_{imp}(K, T)$ generated by the model with $\sigma = 0.50, \beta = 1.00, \lambda_0 = 50, T = 0.5 yrs., \gamma = 0.03, r = 10\%$. We assume that the stock price is \$100. The effective dividend rate is $d_{imp}(100, T) = 14\%$. Notice that the shape in implied dividend curve is consistent with the curves in Figures 3 and 4 which were derived from market data. For low strikes, the drop in value is related to the early-exercise of calls, a feature unique to HTBs. For high strikes, the broad increase corresponds to the classical early exercise property of in-the-money puts. The values of the parameters (for example, the initial buy-in rate of approximately one per week ($\lambda = 50$) are not atypical for many HTBs.

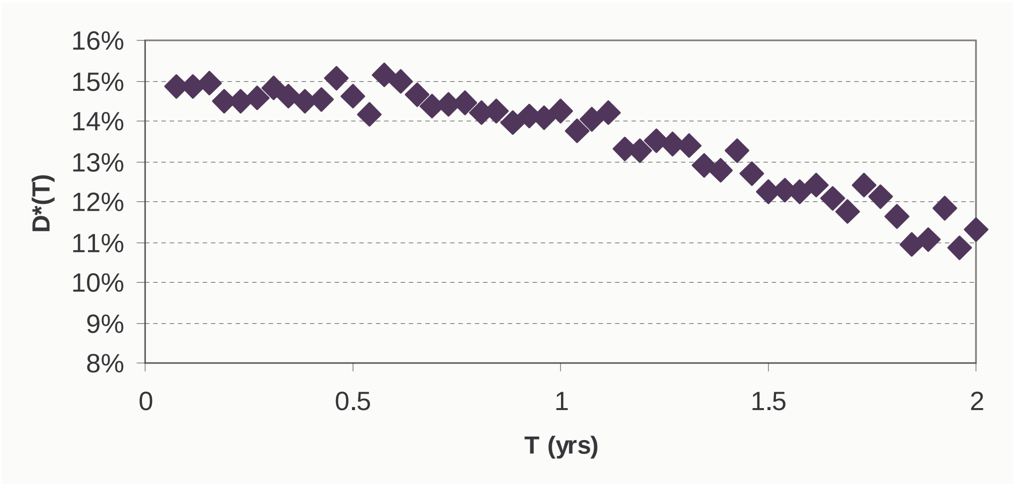

Figure 9: Term-structure of effective dividend rates $D^*(T)$ for the following choice of parameters: $\lambda_0 = 15, \gamma = 0.01, \beta = 30, \sigma = 0.5$

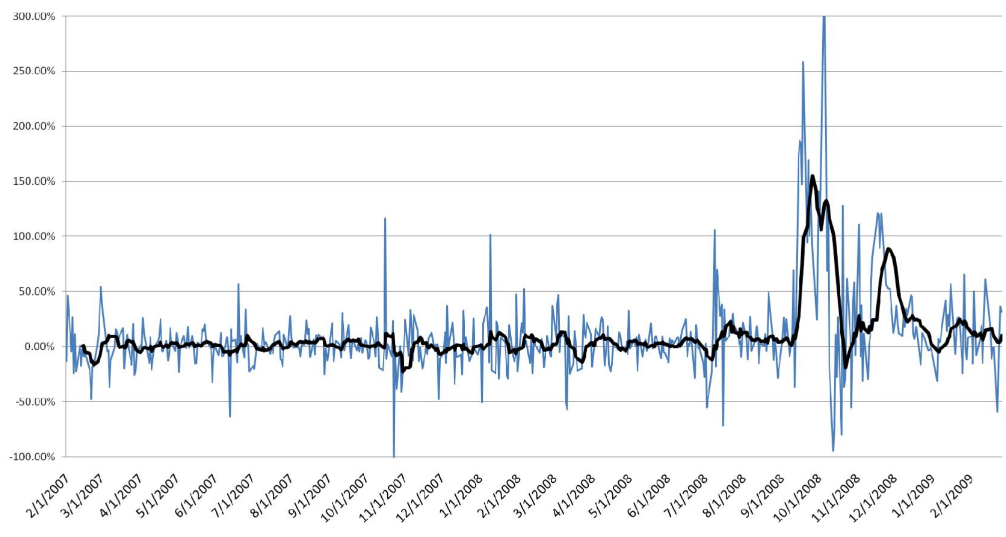

Figure 10: The thin line corresponds to the daily values of the hard-to-borrowness parameter ρ_t, in percentage points, estimated from equation (16). The thick line corresponds to a 10-day moving average of the latter. Smoothing removes noisy effects due to volatility and end-of-day quotes. Notice that the 10-day moving average for the hard-to-borrowness, exceeds 100% in September-October 2008 and that it remains elevated until March 2009.

borrowing the underlying shares should be reflected in the value of the fund. Thus, we should be able to observe the borrowing costs, by comparing the prices of the short-leveraged product with the underlying ETF or with a long-leveraged product.

Let $U_t^{(2)}$ and $U_t^{(-2)}$ denote respectively the prices of a double-long ETF and a double-short ETF on the same underlying index. We shall denote the price of the ETF corresponding to the underlying index by S_t. It follows from the definition of these products that the daily prices changes should follow the equations

$$\frac{dU_t^\beta}{U_t^\beta} = \beta \frac{dS_t}{S_t} - \beta(r - \rho_t)dt + rdt - fdt, \quad (\beta = +2, -2), \tag{13}$$

where r is the benchmark funding rate (LIBOR or Fed Funds), f is the expense ratio or management fee and

$$\begin{aligned} \rho_t &= \gamma\lambda_t \ \text{ if } \beta < 0, \\ &= 0 \ \text{ if } \beta > 0. \end{aligned} \tag{14}$$

Thus ρ_t is the instantaneous (annualized) rent that is associated with shorting the underlying stock. We can view ρ_t as a proxy for $\gamma\lambda_t$, the expected shortfall for a short-seller subject to buy-in risk, or the "fair" reduced rate associated with shorting the underlying asset.

Using the above equations with $\beta = 2$ and $\beta = -2$, we obtain

$$\frac{dU_t^{(2)}}{U_t^{(2)}} + \frac{dU_t^{(-2)}}{U_t^{(-2)}} = 2\left((r - \rho_t) - f\right) dt$$

which implies that

$$\rho_t \; dt = \frac{\frac{dU_t^{(2)}}{U_t^{(2)}} + \frac{dU_t^{(-2)}}{U_t^{(-2)}} + (2f - 2r)\, dt}{-2}. \tag{15}$$

This suggests that we can use daily data on leveraged ETFs to estimate ρ_t.

For the empirical analysis, we used dividend-adjusted closing prices from the PowerShares Ultrashort Financial ETF (SKF) and the PowerShares Ultra long Financial ETF (UYG). The underlying ETF is the Barclays Dow Jones Financial Index ETF (IYF). Using historical data, we computed the right-hand side of equation (15), which we interpret as corresponding to daily sampling, with $dt = 1/252$, $r =$3-month LIBOR and $f = 0.95\%$, the expense ratio of SKF and UYG advertised by Powershares. The results of the simulation are seen in Figure 10.

We see that ρ_t, the cost of borrowing, varies in time and can change quite dramatically. In Figure 10, we consider a 10-day moving average of ρ_t to smooth out the effect of volatility and end-of-day marks. The data shows that increases in borrowing costs, as implied from the leveraged ETFs, begin in the late summer of 2008 and intensified in mid-September, when Lehman Brothers collapsed and the SEC ban on shorting 800 financial stocks was implmented (the latter occurred on September 19, 2008). Notice that the implied borrowing costs for financial stocks remain elevated subsequently, despite the fact that the SEC ban on shorting was removed in mid-October. This calculation may be interpreted as exhibiting the variations of λ_t (or $\gamma\lambda_t$) for a basket of financial stocks. For intance, if we assume that the elasticity γ remains constant (e.g. at 2%), the buy-in rate will range from a low number (e.g $\lambda = 1$, or one buy-in per annum) to 50 or 80, corresponding to several buy-ins per week.

6 Conclusions

In the past, attempts have been made to understand option pricing for HTB stocks with models that do not take into account price-dynamics. The latter approach leads to a view of put-call parity which is at odds with the functional equilibrium (steady state) evidenced in the options markets, in which put and call prices are stable and yet "naive" put-call parity does not hold. The point of this paper has been to fix this and show how dynamics and pricing are intertwined. The notion of effective dividend is the principal consequence of our model as far as pricing is concerned. We also obtain a term-structure of dividend yields. Reasonable parametric choices lead to a term-structure which is concave down, a shape frequently seen in real option markets. The model also reproduces the (American) early exercise features, including early exercise of calls, which cannot happen for non-dividend paying stocks which are easy to borrow.

Consequences of our model for dynamics are elevated volatilities, sharp price spikes and occasional crashes followed by often dramatically lower hard-to-borrowness. Finally, we point out that short-selling restrictions are a prominent feature of many developing markets (e.g., India, China, Brazil) and are also encountered in G7 markets. In forthcoming publications, we shall study shorting costs via this model in other situations. In particular, two areas that seem to provide interesting costs are the A-shares and H-shares in China (where A-shares cannot be shorted). Another forthcoming paper will study leveraged ETFs in more detail.

Acknowledgement. We thank the referees for many helpful and insightful comments. Thanks also to Sacha Stanton of Modus Inc. for assistance with options data and Stanley Zhang for exciting discussions on leveraged ETFs.

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