First-passage time of BM from a strip

Let $X(t)$ be a standard Wiener process (Brownian motion with variance 1 and drift 0). We consider the "strip" $\{(x,t):(x,t)\in(a,b)\times(0,\infty)\}$ and ask for the probability that BM does not exit the strip in time T. More generally, let $F(x)$ be a function defined for $a<x<b$. We seek a closed-form solution for the expectation value

$$E\left\{a<min_{t<T}X(t)<max_{t<T}X(t)<b;F(X(T))\right\}. \tag{1}$$

If $F(x)=1$ then this represents the probability that BM is confined to the strip up to time T.

We shall use the "method of reflections" to derive the Green function associated to this calculation. If there is no band,i.e. in the case of free BM, then

$$\begin{aligned} E\left\{F(X(T)|X(t)=x\right\} &= \frac{1}{\sqrt{2\pi(T-t)}}\int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2(T-t)}}F(y)dy \\ &= \int_{-\infty}^{\infty} G_0(x,t;y,T)F(y)dy \end{aligned} \tag{2}$$

with $G_0(x,t;y,T)$, the Green function for the standard diffusion equation, given by

$$G_0(x,t,y,T)=\frac{1}{\sqrt{2\pi(T-t)}}e^{-\frac{(y-x)^2}{2(T-t)}}.$$

We seek an analogous representation for the solution of (1) of the form

$$\int_a^b G(x,t;y,T)F(y)dy \tag{3}$$

where $G(x,t;y,T)$ needs to be determined. The idea is to use the method of "image sources" in Physics a.k.a. the Reflection Principle.

1 The case of one barrier ($b=+\infty$)

In this case, we need a function $G(x,t;y,T)$ which satisfies the associated diffusion equation

$$\frac{\partial G(x,t;y,T)}{\partial t}+\frac{1}{2}\frac{\partial^2 G(x,t;y,T)}{\partial x^2}=0$$

and vanishes on $x=a$. For this, we try

$$G_1(x,t;y,T) = G_0(x,t;y,T) - G_0(x,t;2a-y,T) \tag{4}$$

Clearly, this new function satisfies the PDE in x, t and vanishes when $x = a$. The intuition is that $2a - y = y^*$ is the "reflected source" which is at the same distance to a as y. Therefore, by symmetry, the difference vanishes at $x = a$.

In particular, this implies that

$$Prob\{\tau_a > T | X_t = x\} = \int_a^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(y-x)^2}{2(T-t)}} dy - \int_a^\infty \frac{1}{\sqrt{2\pi(T-t)}} e^{-\frac{(2a-y-x)^2}{2(T-t)}} dy$$

and hence that

$$\begin{aligned} Prob\{\tau_a > T | X_0 = 0\} &= \int_a^\infty \frac{1}{\sqrt{2\pi T}} e^{-\frac{y^2}{2T}} dy - \int_a^\infty \frac{1}{\sqrt{2\pi T}} e^{-\frac{(2a-y)^2}{2T}} dy \\ &= \frac{2}{\sqrt{2\pi T}} \int_0^{|a|} e^{-\frac{(y)^2}{2T}} dy = ERF(|a|/\sqrt{2T}) \end{aligned} \tag{5}$$

2 Two barriers

We construct the Green's function iteratively. Let

$$G_1(x,t;y,T) = G_0(x,t;y,T) - G_0(x,t;2a-y,T)$$

This function vanishes for $x = a$ but not for $x = b$. To make it vanish there, we reflect y and $2a - y$ around b, to obtain

$$\begin{aligned} G_2(x,t;y,T) &= G_0(x,t;y,T) - G_0(x,t;2a-y,T) - (G_0(x,t;2b-y,T) - G_0(x,t;2b-2a+y,T)) \\ &= G_1(x,t;y,T) - G_1(x,t;2b-y,T). \end{aligned} \tag{6}$$

Clearly, G_2 satisfies the correct boundary conditions at $x = b$ but not at $x = a$. We can remedy this by reflecting it again around $x = a$. If we do this, this affects only the terms in parenthesis, since G_1 satisfies the BC. Accordingly, we obtain

$$\begin{aligned} G_3(x,t;y,T) &= G_0(x,t;y,T) - G_0(x,t;2a-y,T) - (G_0(x,t;2b-y,T) - G_0(x,t;2b-2a+y,T)) \\ &+ G_0(x,t;2a-2b+y,T) - G_0(x,t;2a-2b+2a-y,T). \end{aligned} \tag{7}$$

This procedure is iterated to all orders, cancelling successively the values at a and b.

First, we note that a pattern emerges in terms of the "sources": after summing up, we obtain a concise formula (see Feller, *Introd. to Probability Theory and its Applications*, Vol 2),

$$G(x,t;y,T) = \sum_{n=-\infty}^{+\infty} \left(G_0(x,t;y+2nc,T) - G_0(x,t;2a-y-2nc,T) \right) \qquad (8)$$

To obtain this equation, we simply examine all the terms with positive sign in (7) and all terms with negative sign in (7), and verify that they correspond to the source locations proposed in (8). Equation (8) represents the Green function for the heat equation on the strip $(a,b) \times (0,T)$.

To calculate the probability of survival, we set $x = t = 0$ and calculate the series in terms of the cumulative normal distribution function (NORMDIST), using (5) as guide. Accordingly, (with $c = b - a$),

$$Prob\left\{\tau_{ab} < T | X_0 = 0\right\} = \sum_{n=-\infty}^{+\infty} \int_a^b \left(G_0(0,0;y+2nc,T) - G_0(0,0;2a-y-2nc,T) \right) dy$$

$$\begin{aligned} = & \sum_{n=-\infty}^{+\infty} [NORMDIST(b+2c,0,T,true) - NORMDIST(b-2a+2nc,0,T,true) \\ + & \; NORMDIST(a+2nc,0,T,true) - NORMDIST(-a+2nc,0,T,true)] \end{aligned} \qquad (9)$$

Exercise: Verify the validity of (8) by induction. Build a spread sheet for calculating the above probability in Excel, using a partial sum as an approximation to the series. The inputs should be the endpoints a and b and the time horizon T.