Derivative Securities: Lecture 7

Interest rate swaps and swaptions

Sources:
Instructor notes
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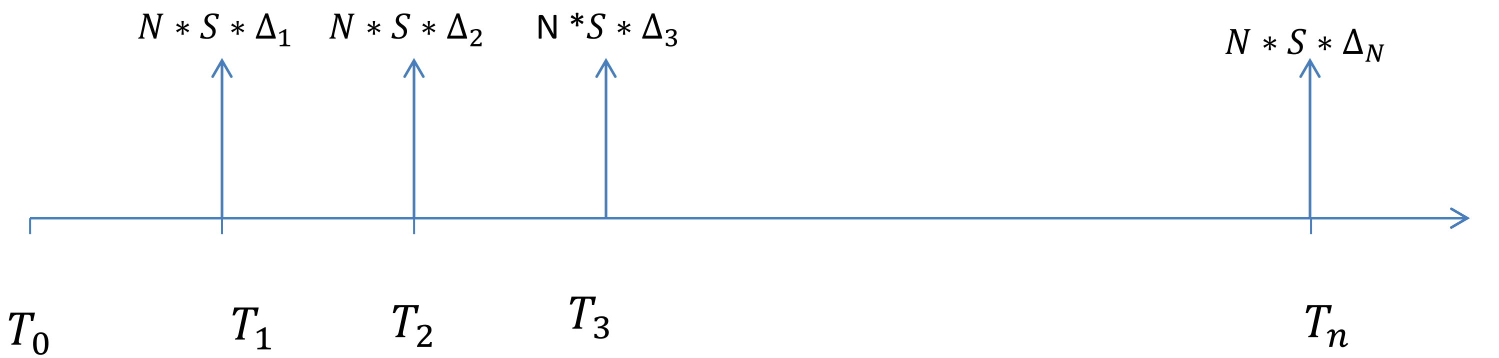
Interest Rate Swaps

- Swaps are among the most traded derivatives
- In some contexts (e.g. regulatory) the expressions ``swaps'' and ``derivatives'' are used interchangeably
- In a plain-vanilla swaps, two counterparties exchange cash-flows periodically at fixed dates. The most classical swap is structured as followed

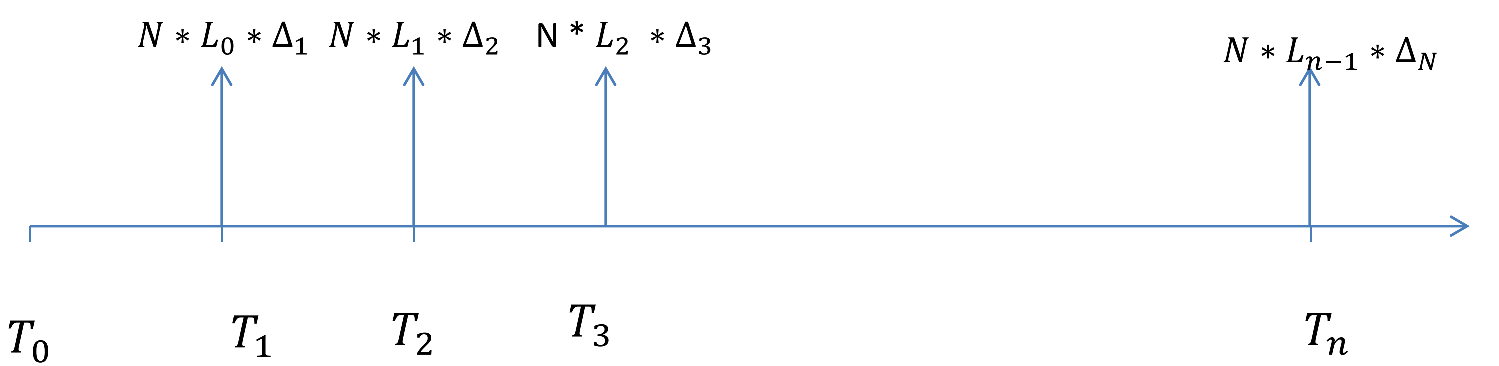
  (a) notional amount (N)
  (b) cash-flow dates $T_1 T_2, \ldots., T_n$
  (c) settlement date T_0
  (d) fixed rate (S)
  (e) floating rate (e.g. Libor 3m)

Cash-flow structure

Fixed-rate leg:



Floating-rate leg:



Comments & Pricing

- The floating rate is usually set **in arrears**
- The annualized time intervals are determined by a day-count convention, typically ``actual/360'' for IR swaps

  Example: if there are 181 days between two (semiannual) payment dates, T_{a-1},T_a, then Δ_a=$\frac{181}{360}$.
- Most swaps are traded in such a way that the floating and fixed legs have the same value, which means that the swap ``price'' would be zero.
- Pricing swaps is done by discounting cash-flows, using, for example, the discount curve, $D(t)$, where

Valuing Swaps

If $T_{j-1} <$ t $< T_j$

$$FixedLeg(t) \ = N \sum_{i:T_i > t} S\Delta_i D(T_i)$$

$$FloatingLeg(t) = N - N \frac{t - T_{j-1}}{T_j - T_{j-1}} \Delta_j L_{j-1}$$

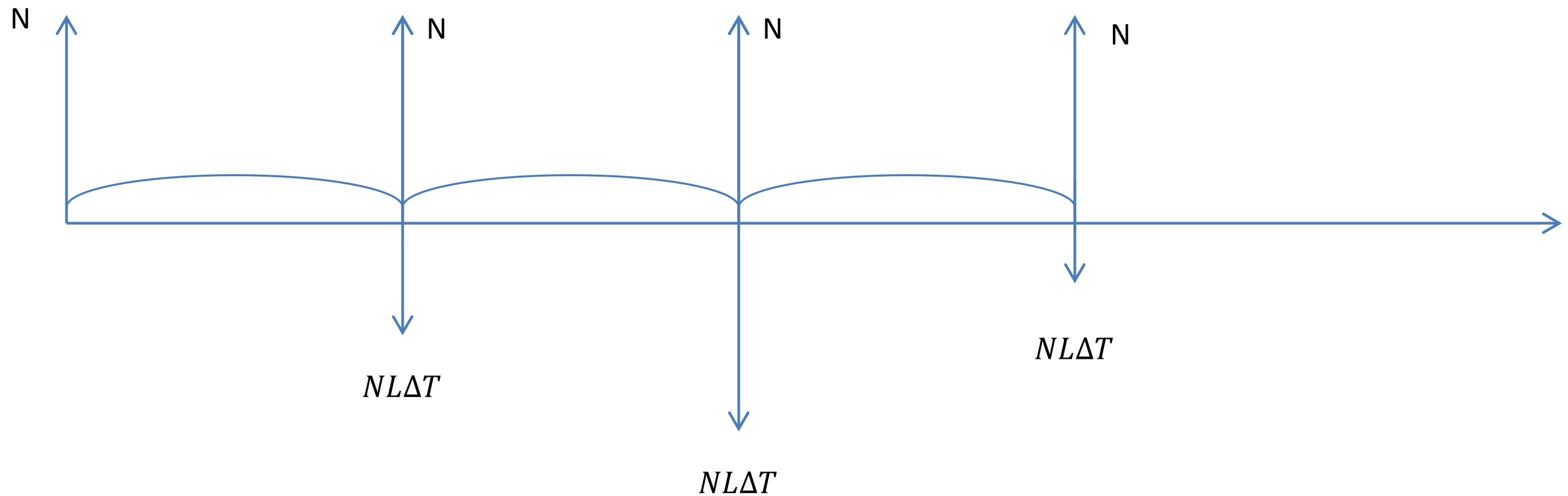
If $t = T_j$+

$$FixedLeg(t) = N \sum_{i:T_i > t} S\Delta_i D(T_i)$$

$$FloatingLeg(t) = N$$

Floating leg value =Par

If $t = T_j+$, we can replicate the cash-flows of the floating leg by borrowing \$N and investing the proceeds in the Libor market and paying the Floating Rate for the period



This assumes that the swap's floating rate is exactly the Libor rate for each period. However, the most liquid rate is the 3M Libor and the most common payment frequency is semi-annual.

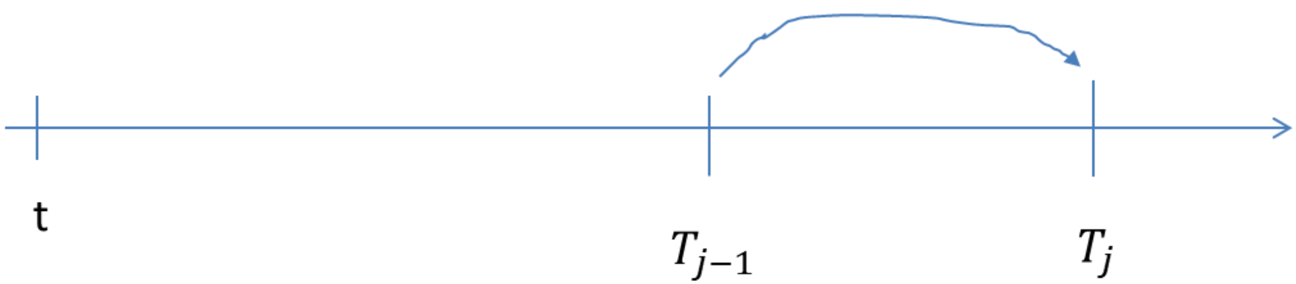
Discount Factors and Forward Rates

$$D(T_i) = \frac{1}{(1 + Libor(t, T_i) * (T_i - t))} = \prod_j \frac{1}{(1 + FR_{j-1,j}\Delta_j)}$$

- Libor term rates (more than 6 M) can be observed or derived from ED futures.

- Forward rates are estimated using ED futures for short maturities (convexity adjustment)

- Longer forward rates are generated by ``bootstrapping''



``Bond interpretation''

- For the fixed-rate payer, a swap is equivalent to being long a Floating Rate Bond and short a fixed coupon bond with same cash-flow dates.

Or

$$\boxed{\text{Bond + (fixed-rate payer swap) = Floating rate Bond}}$$

FRA Interpretation

- A swap is a portfolio of forward rate agreements in which the fixed-rate payer will borrow \$N dollars for N consecutive periods at a fixed rate S.

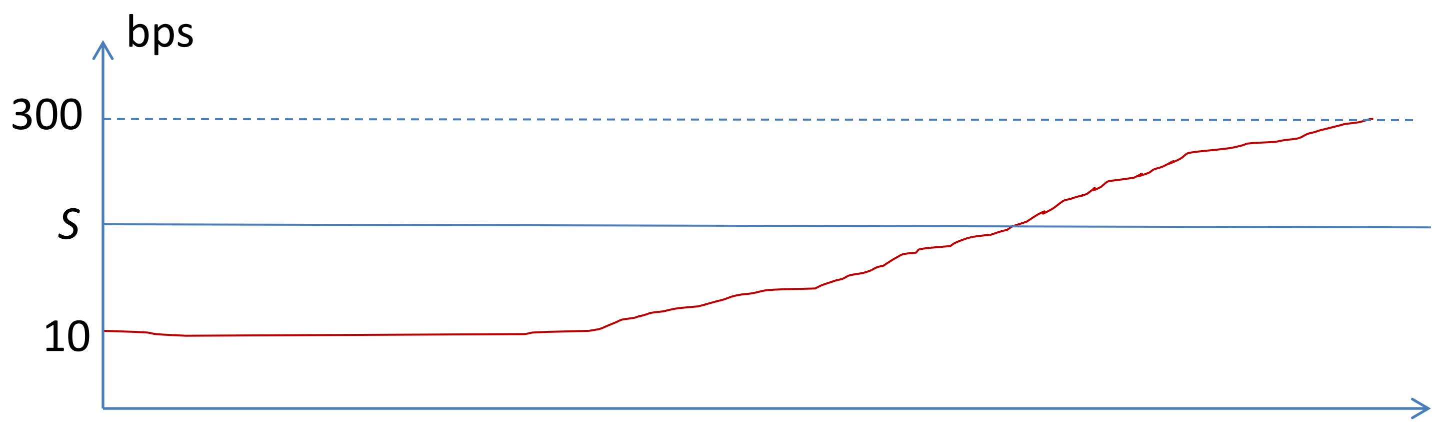
- Accordingly, the value of the swap for the fixed rate payer is

$$V = N \sum_{T_i > t} \left(FR_{i-1,i} - S \right) \Delta_i D(T_i)$$

$$S_{atm} = \frac{\sum_{T_i > t} FR_{i-1,i} \Delta_i D(T_i)}{\sum_{T_i > t} \Delta_i D(T_i)}$$

$$\boxed{S_{atm} = \frac{D(T_n) - D(t)}{\sum_{T_i > t} \Delta_i D(T_i)}}$$

Valuation using Forward Rate Curve



- The swap rate is an average of FR

- If the FR curve is upward sloping (normal) then fixed payers are OTM for payments and in the money for later payments

  Fixed payers pay more than they receive at the beginning of the swap and expect to ``catch up''.

  Fixed payers are long a ``basket of rates''; fixed receivers are short the basket.

Swap risk measures

- DV01= ``Dollar value of a basis point'' refers to the exposure of a swap position to a move of 1 bps in the forward rate curve.

Use bond interpretation: fixed-rate receiver is long a bond with coupon S, short a floater.

Floater has no risk; therefore

$$DV01 = N \frac{d}{dr}\left(\sum_{i=1}^{N} S\Delta_i\, D(T_i) + D(T_n)\right) 10^{-4}$$

$$= N \frac{d}{dr} B(S,T_n)\, 10^{-4}$$

$$\cong - N\, B(S,T_n)\;\; D(S,T_n)\, 10^{-4}$$

$D(S,T)$=- $\frac{\frac{d}{dr}B(S,T_n)}{B(S,T_n)}$=``duration''

Bond Math

$$B(C,\Delta,Y,n) = \sum_{i=1}^{n} \frac{\Delta C}{(1+\Delta\cdot Y)^i} + \frac{1}{(1+\Delta\cdot Y)^n}$$

$$\frac{d}{dY}B(C,\Delta,Y,n) = -\frac{1}{1+\Delta\cdot Y}\left(\sum_{i=1}^{n} \frac{\Delta C i \Delta}{(1+\Delta\cdot Y)^i} + \frac{n\Delta}{(1+\Delta\cdot Y)^n}\right)$$

$$= -\frac{1}{1+\Delta\cdot Y}\left(\sum_{i=1}^{n} \frac{\Delta C T_i}{(1+\Delta\cdot Y)^i} + \frac{T_n}{(1+\Delta\cdot Y)^n}\right)$$

$$D(C,\Delta,Y,T) = -\frac{\frac{d}{dY}B(C,\Delta,Y,n)}{B(C,\Delta,Y,n)} = \frac{\left(\sum_{i=1}^{n}\frac{\Delta C T_i}{(1+\Delta\cdot Y)^i} + \frac{T_n}{(1+\Delta\cdot Y)^n}\right)}{\sum_{i=1}^{n}\frac{\Delta C}{(1+\Delta\cdot Y)^i} + \frac{1}{(1+\Delta\cdot Y)^n}}$$

$$= ``Average\ time'' = Duration$$

DV01 for ATM swap

$$DV01 = ND(S,T)10^{-4}$$

$$N = \frac{DV01}{Duration}10^{4}$$

- What is the notional amount for a T year swap which gives me a 1MM USD DV01 exposure?

Answer=$\frac{10^{10}}{Duration}$=$\frac{USD\ 10\ billion}{Duration}$

If we assume that the duration of a 30-year swap is 15 years, then the notional amount corresponding to 1million DVO1 is 10/15 billion = 666 million dollars

DV01 exposure for a ED Futures & Application

1 ED has 25 USD variation per basis point move in rates

The ED contract mimics a loan for 1MM for 3 months at Libor rate.

Change in contract for 1 basis point change is 0.25*0.0001*1,000,000=25

Application:

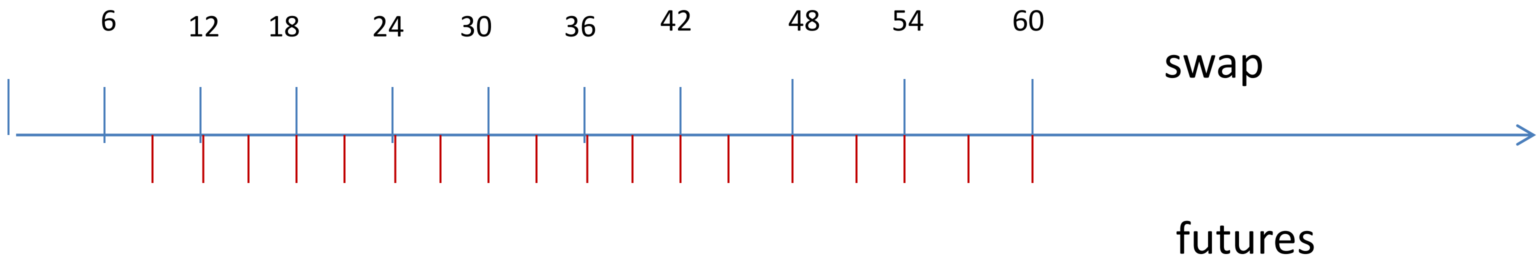
Hedge a 5 year ATM swap fixed-rate payer, assuming 1MM DV01, with ED futures

A 1 mm DV01 exposure for an ED corresponds to **40,000** contracts.

Buy a strip of 18 futures (4 per year, first 2 not needed),
with 40,000/18 ≅2,222 contracts per maturity

(Typical volume on CME ~ 100K contracts/day)

Picture of the hedge



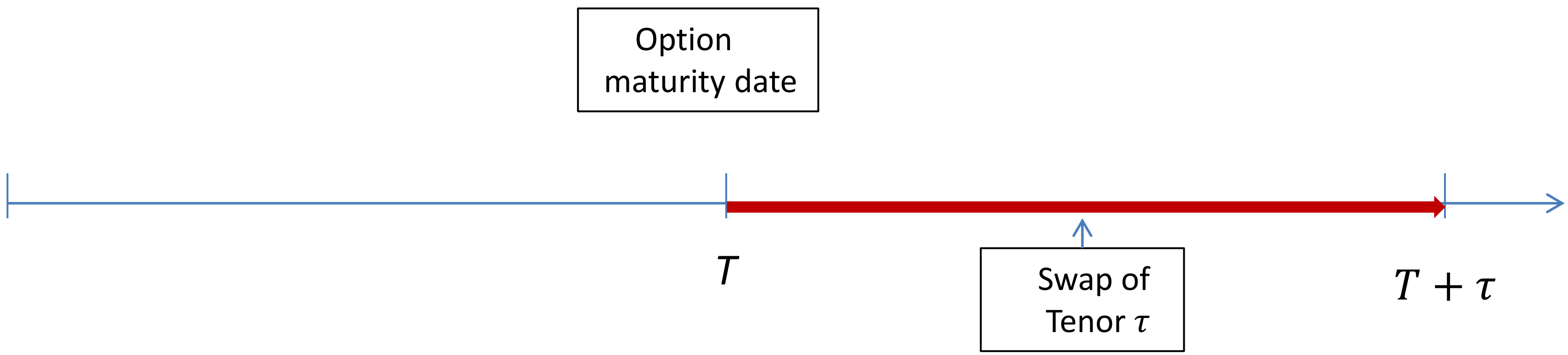
- Spreading the hedge evenly is necessary because the interest-rate curve does not move exactly in parallel.

- This approach mimics the actual floating leg cash-flows of the swap

Swaptions

- A payer swaption is an option to enter into a swap at a later date, paying fixed rate.
- A receiver swaption is an option to enter into a swap at a later date, receiving fixed.
- Payer swaption: ``call on forward swap rate''
- Receiver swaption: ``put on forward swap rate''
- **Bermudan swaptions:** can be exercised on swap cash-flow dates (American)
- Motivation for swaptions: swaptions are used to hedge issuance of bonds or to hedge call features in bonds (typically in FNMA and other Agencies, for hedging forward rate exposures, etc).

Valuation of Swaptions



<u>Payer swaption:</u>

Notional (N)
Maturity (T)
Tenor (τ)
Strike (K)

At maturity date, the payer swap exercises
If the swap rate is higher than the strike rate

Value of the swaption on date T :

$$max\left(N\sum_{i=1}^{n}(S_{atm}(T)-K)\Delta_i D(T,T_i),0\right)$$

Swaption Valuation

Payer payoff

$$max\left(N\sum_{i=1}^{n}(S_{atm}(T)-K)\Delta_i D(T,T_i),0\right)=\left(N\sum_{i=1}^{n}D(T,T_i)\Delta_i\right)max(S_{atm}(T)-K,0)$$

Receiver payoff

$$\left(N\sum_{i=1}^{n}D(T,T_i)\Delta_i\right)max(K-S_{atm}(T),0)$$

Long payer/short receiver

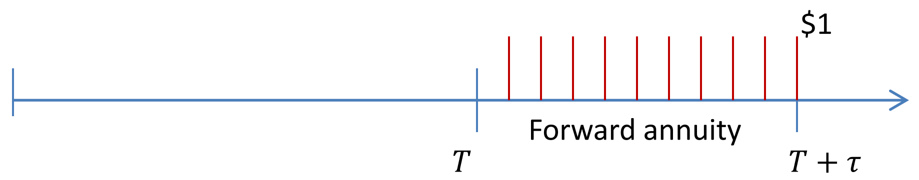
$$\left(N\sum_{i=1}^{n}D(T,T_i)\Delta_i\right)(S_{atm}(T)-K)$$

But… ATM swap should
Cost zero to get into

Put Call Parity For Swaptions

$$Pay(K,T,\tau) - Rec(K,T,\tau) = \left(N \sum_{i=1}^{n} \Delta_i \, D(0,T_i) \right) (FS(T) - K)$$

- Here, the term in parenthesis is the present value of a forward annuity



- $FS(T)$ is the forward swap rate for settlement at time T of a swap with tenor τ.

Swaption pricing a la Arrow Debreu

- In different states of the world I get different spot rates S_{atm}(T).
- In each state, the swaption is worth either zero or an annuity with coupon $S_{atm}(T) - \mathrm{K}$. Therefore, the value of a swaption should be

$$Pay(K,T,\tau) = D(0,T)E\left\{\left(N\sum_{i=1}^{n} D(T,T_i)\Delta_i\right) max(S_{atm}(T) - K, 0)\right\}$$

The expectation is consistent with PCP so we must have

$$D(0,T)E\left\{\left(N\sum_{i=1}^{n} D(T,T_i)\Delta_i\right)(S_{atm}(T) - K)\right\}$$
$$= \left(N\sum_{i=1}^{n} \Delta_i\, D(0,T_i)\right)(FS(T) - K)$$

For all K

Characterization of the Expectation

We have

$$D(0,T)E\left\{\left(\sum_{i=1}^{n} D(T,T_i)\Delta_i\right)S_{atm}(T)\right\} = \left(\sum_{i=1}^{n} \Delta_i\, D(0,T_i)\right)FS(T)$$

Or, finally

$$E\left\{\left(\sum_{i=1}^{n} D(T,T_i)\Delta_i\right)S_{atm}(T)\right\} = \left(\sum_{i=1}^{n} \Delta_i \frac{D(0,T_i)}{D(0,T)}\right)FS(T)$$

Another constraint

This expectation operator also can be used for pricing forward annuities and we have

$$E\left\{\left(\sum_{i=1}^{n} D(T,T_i)\Delta_i\right)\right\} = \left(\sum_{i=1}^{n} \Delta_i \frac{D(0,T_i)}{D(0,T)}\right)$$

This equation follows from the fact that a forward annuity can be replicated by a series of zero-coupon bonds.

The left hand side is the <u>pricing formula</u>; the right hand side is the <u>replication value</u> using zero-coupon bonds

Reformulation of the Swaption Pricing formula

$$Pay(K,T,\tau) = D(0,T)E\left\{\left(N\sum_{i=1}^{n} D(T,T_i)\Delta_i\right) max(S_{atm}(T) - K, 0)\right\}$$

$$= \frac{\left(N\sum_{i=1}^{n} D(0,T_i)\Delta_i\right)}{E\left\{\left(N\sum_{i=1}^{n} D(T,T_i)\Delta_i\right)\right\}} E\left\{\left(N\sum_{i=1}^{n} D(T,T_i)\Delta_i\right) max(S_{atm}(T) - K, 0)\right\}$$

$$= N\left(\sum_{i=1}^{n} D(0,T_i)\Delta_i\right)\tilde{E}\{max(S_{atm}(T) - K, 0)\}$$

Where

$$\tilde{E}\{\phi\} \coloneqq \frac{E\{A\phi\}}{E\{A\}}$$

The Annuity Measure

$$\tilde{E}(S_{atm}(T)) = FS(T)$$

Assume lognormal spot rate,

$$\boxed{Pay(K,T,\tau) = N\left(\sum_{i=1}^{n} D(0,T_i)\Delta_i\right) BSCall(FS(T,\tau),T,K,0,0,\sigma)}$$