

Derivative Securities: Lecture 4

Introduction to option pricing

Sources:

J. Hull, 7th edition

Avellaneda and Laurence (2000)

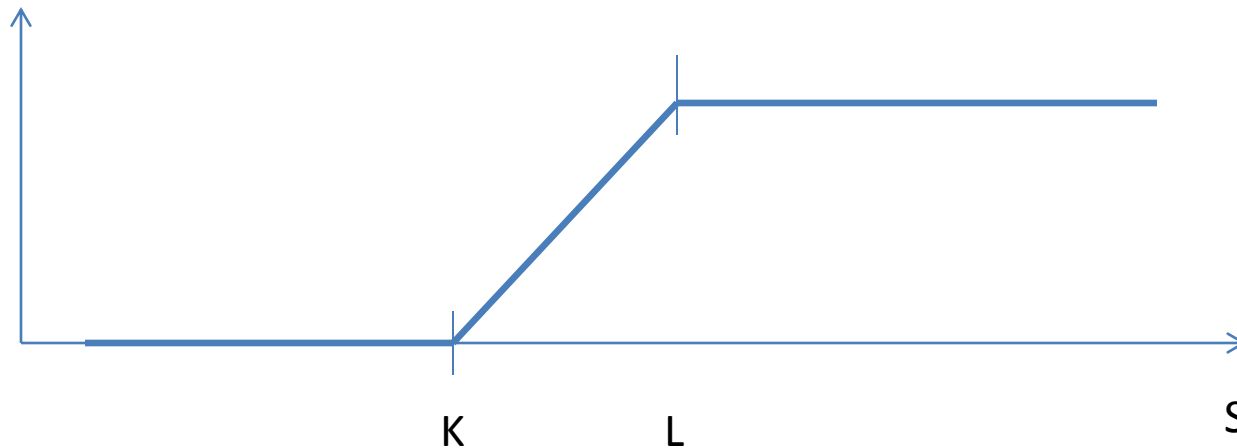
Yahoo!Finance & assorted websites

Option Pricing

- In previous lectures, we covered forward pricing and the importance of cost-of carry
- We also covered Put-Call Parity, which can be viewed as relation that should hold between European-style puts and calls with the same expiration
- Put-call parity can be seen as pricing conversions relative to forwards on the same underlying asset
- What other relations exist between options and spreads on the same underlying asset?

Call Spread

Call Spread: Long a call with strike K, short a call with strike L ($L > K$)



Since the payoff is non-negative, the value of the spread must be positive

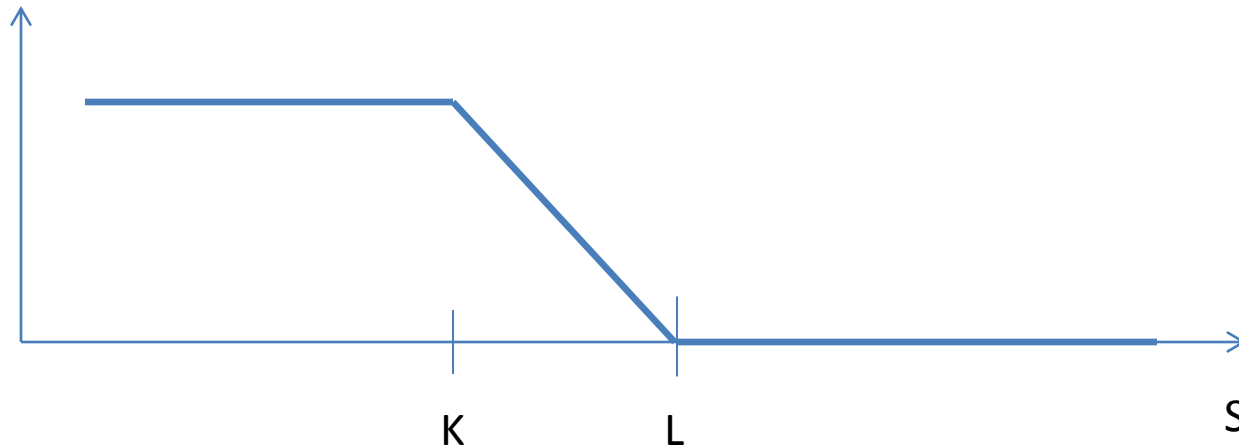
$$K < L \quad \Rightarrow \quad Call(K, T) > Call(L, T)$$

$$CS(K, L, T) = Call(K, T) - Call(L, T) > 0$$

Spread makes money if the price of the underlying goes up

Put Spread

Put Spread: Long a put with strike L, short a put with strike K ($L > K$)



Since the payoff is non-negative, the value of the spread must be positive

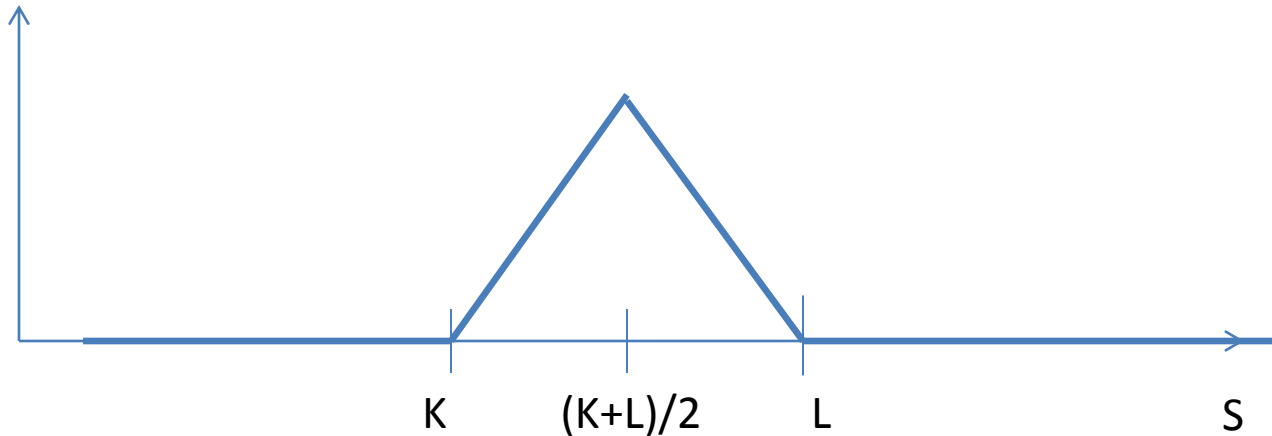
$$K < L \quad \Rightarrow \quad Put(K, T) < Put(L, T)$$

$$PS(K, L, T) = Put(L, T) - Put(K, T) > 0$$

Spread makes money if the price of the underlying goes down

Butterfly Spread

Butterfly spread: Long call with strike K, long call with strike L, short 2 calls with strike $(K+L)/2$



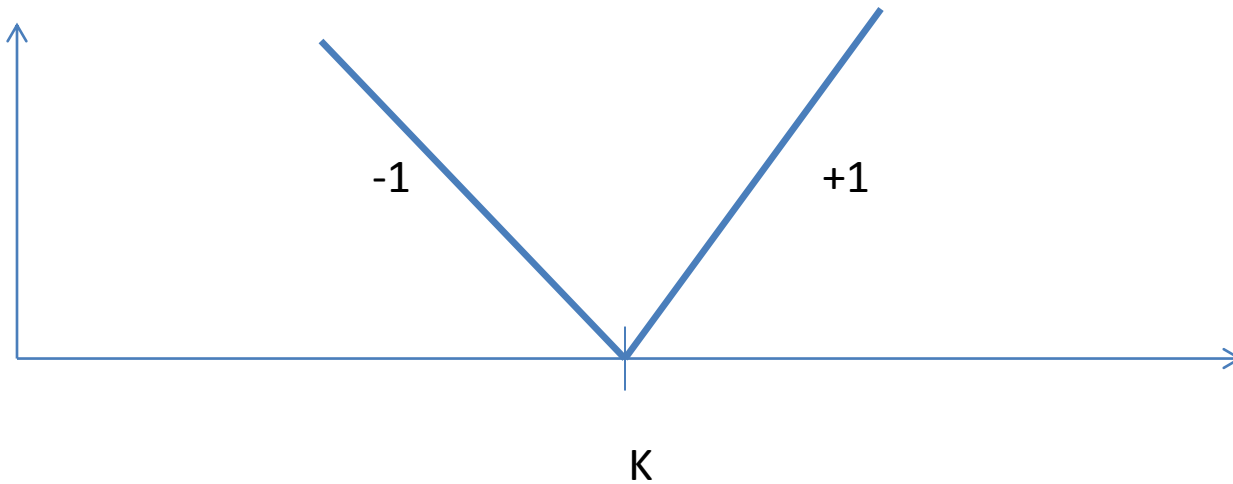
Since the spread has non-negative payoff, it must have positive value

$$B(K, (K + L) / 2, L, T) \equiv Call(K, T) + Call(L, T) - 2Call\left(\frac{K + L}{2}, T\right) > 0$$

Butterflies make \$ if the stock price is near $(K+L)/2$ at expiration.

Straddle

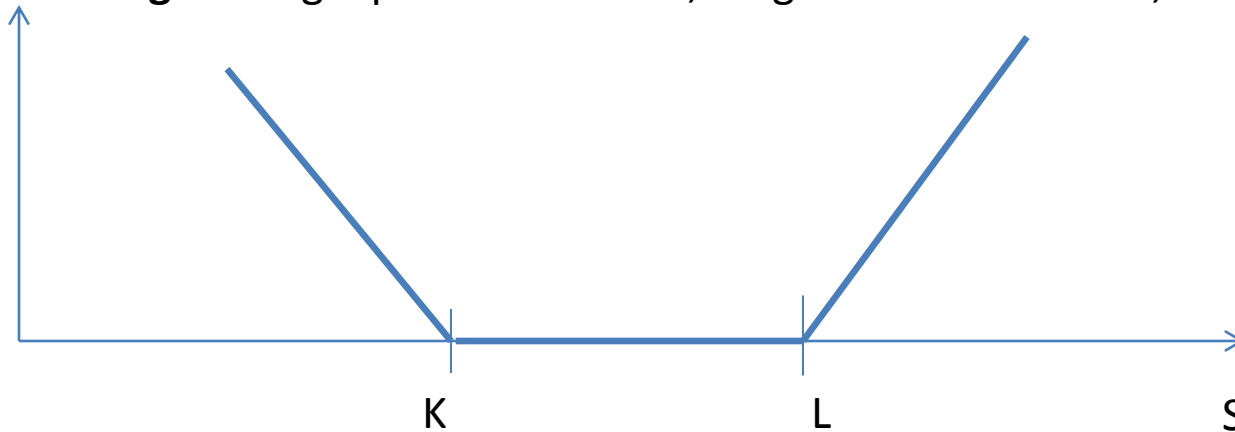
Straddle: Long call and long put with the same strike



Straddles make money if the stock price moves away from the strike and ends far from it

Strangle

Strangle: Long 1 put with strike K , long 1 call with strike L , $L > K$.

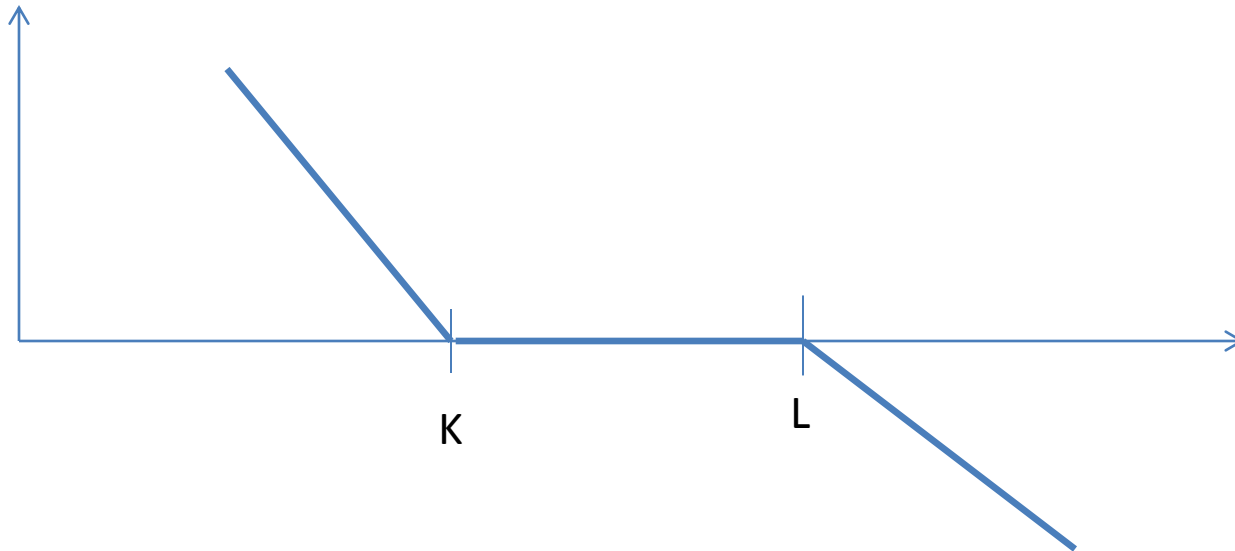


Strangle is also non-directional, like a straddle, but makes \$ only if the stock moves very far away.

Straddles and strangles are often used to express views about volatility of the underlying stock and are non-directional.

Risk-reversal

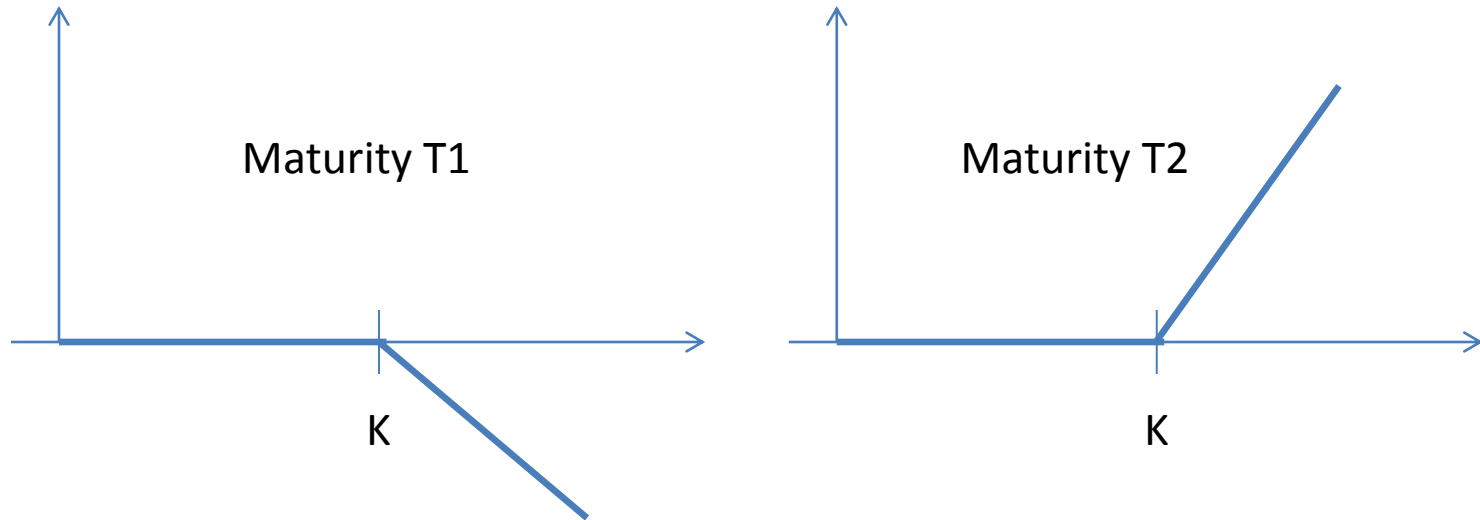
Risk-reversal : Long 1 put with strike K , short 1 call with strike L , $L > K$.



Directional spread. Can be seen as financing a put by selling an upside call.

Calendar Spread

Calendar Spread: Short 1 call with maturity T_1 , Long 1 call with maturity T_2 , $T_1 < T_2$
Same strike

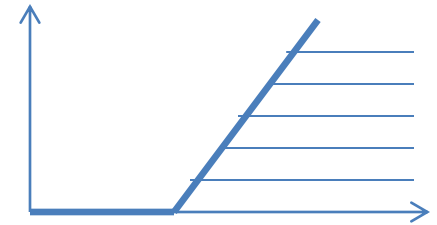


- If the underlying pays no dividends between T_1 and T_2 , then the longer maturity call is above intrinsic value at time T_1 . Calendars have positive value.
- For American options, calendars always have positive value

Reconstructing Call prices from Butterfly Spreads

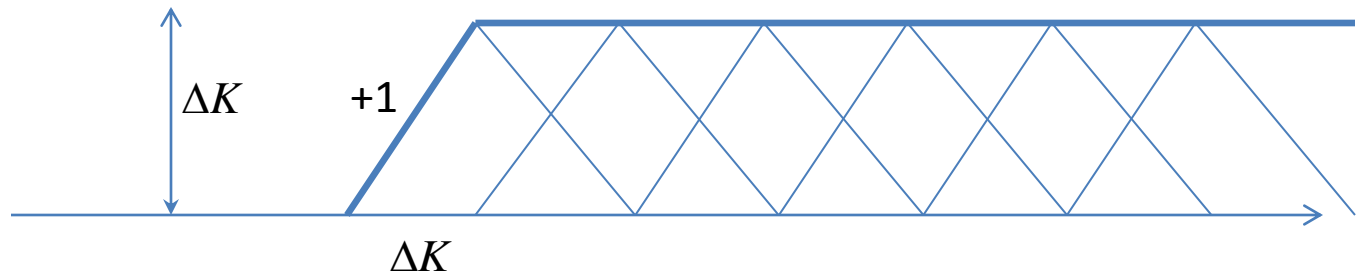
Assume for simplicity a countable number of strikes, $K_n = n\Delta K$, $n = 0, 1, 2, 3, \dots$
 and that the stock price can only take values on the lattice $S_T = m\Delta K$, $m = 1, 2, 3, \dots$

$$Call(K_n, T) = \sum_{j \geq n} (Call(K_j, T) - Call(K_{j+1}, T)) = \sum_{j \geq n} CS(K_j, K_{j+1}, T)$$



- A call can be viewed as a portfolio of call spreads
- A call spread can be viewed as a portfolio of butterfly spreads

$$CS(K_j, K_{j+1}, T) = \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T)$$



Calls as super-positions of butterfly spreads

$$Call(K_n, T) = \sum_{j=n}^{\infty} \sum_{i \geq j} B(K_i, K_{i+1}, K_{i+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) B(K_j, K_{j+1}, K_{j+2}, T)$$

$$= \sum_{j=n}^{\infty} (j+1-n) \Delta K \cdot \left(\frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K} \right)$$

$$= \sum_{j=n}^{\infty} (K_{j+1} - K_n) w(K_{j+1}, T)$$

$$w(K_{j+1}, T) \equiv \frac{B(K_j, K_{j+1}, K_{j+2}, T)}{\Delta K}$$

$$= \sum_{j=0}^{\infty} (K_{j+1} - K_n)^+ w(K_{j+1}, T)$$



The weights correspond to values of Butterfly spreads centered at each K_j . In particular, they are positive

From weights to probabilities

$$w(K, T) > 0$$

$$\sum_1^{\infty} w(K_j, T) = \frac{1}{\Delta K} \sum_1^{\infty} B(K_{j-1}, K_j, K_{j+1}, T) = \frac{1}{\Delta K} CS(0, \Delta K, T)$$

$$= \text{PV}(\$1) = e^{-rT} \quad (\text{assuming that the stock can only take values } > \Delta K)$$

$$w(K_j, T) = e^{-rT} p(K_j, T); \quad \sum_j p(K_j, T) = 1, \quad p(K_j, T) > 0.$$

$$\begin{aligned} \text{Call}(K, T) &= e^{-rT} \sum_j \max(K_j - K, 0) p(K_j, T) \\ &= e^{-rT} E^p \{ \max(S_T - K, 0) \} \end{aligned}$$

First moment of p is the forward price

$$\begin{aligned} E^p \{S\} &= e^{rT} \text{Call}(0, T) \\ &= e^{rT} e^{-qT} S_0 = S_0 e^{(r-q)T} \\ &= F_{0,T} \end{aligned}$$

- A call with strike 0 is the option to buy the stock at zero at time T
Its value is therefore the present value of the forward price (pay now, get stock later).
- It follows that the first moment of p is the forward price.
- It also follows that put prices are given by a similar formula, namely

$$\begin{aligned} \text{Put}(K, T) &= \text{Call}(K, T) + Ke^{-rT} - Se^{-qT} \\ &= e^{-rT} E^p \left\{ (S_T - K)^+ \right\} + e^{-rT} K - e^{-rT} F \\ &= e^{-rT} E^p \left\{ (S_T - K)^+ \right\} + e^{-rT} E^p \{K - S_T\} \\ &= e^{-rT} E^p \left\{ (K - S_T)^+ \right\} \end{aligned}$$

General Payoffs

- Any twice differentiable function $f(S)$ can be expressed as a combination
- of put and call payoffs, using the formula

- $$f(S) = f(0) + f'(0)(S - F) + \int_F^S (S - Y) f''(Y) dY \quad (\text{this is just Taylor expansion})$$

$$= f(0) + f'(0)(S - F) + \int_0^F \underline{(Y - S)^+} f''(Y) dY + \int_F^\infty \underline{(S - Y)^+} f''(Y) dY$$

- Thus, a European-style payoff can be viewed as a spread of puts and calls. By linearity of pricing,

Fair value of a claim with payoff $f(S_T) =$

$$= e^{-rT} f(0) + 0 + \int_0^F \text{Put}(Y, T) f''(Y) dY + \int_F^\infty \text{Call}(Y, T) f''(Y) dY$$

$$= e^{-rT} f(0) + e^{-rT} f'(0) E^P \{S_T - F\} + e^{-rT} E^P \left\{ \int_0^F (Y - S_T)^+ f''(Y) dY + \int_F^\infty (S_T - Y)^+ f''(Y) dY \right\}$$

$$= e^{-rT} E^P \{f(S_T)\}$$

Fundamental theorem of pricing (one period model)

- An **arbitrage opportunity** is a portfolio of derivative securities and cash which has the following properties:
 - The payoff is non-negative in all future states of the market
 - The price of the portfolio is zero or negative (a credit)

Assume that each security has a unique price (i.e. assume bid-offer).

- If there are no arbitrage opportunities, then there exists a probability distribution of future states of the market such that, for any function $f(S)$, the price of a security with payoff $f(S_T)$ is

$$P_f = e^{-rT} E^p \{f(S_T)\}$$

- Conversely, if such a probability exists there are no arbitrage opportunities

Practical Application to European Options

- A pricing measure is a probability of future prices of the underlying asset with the property that

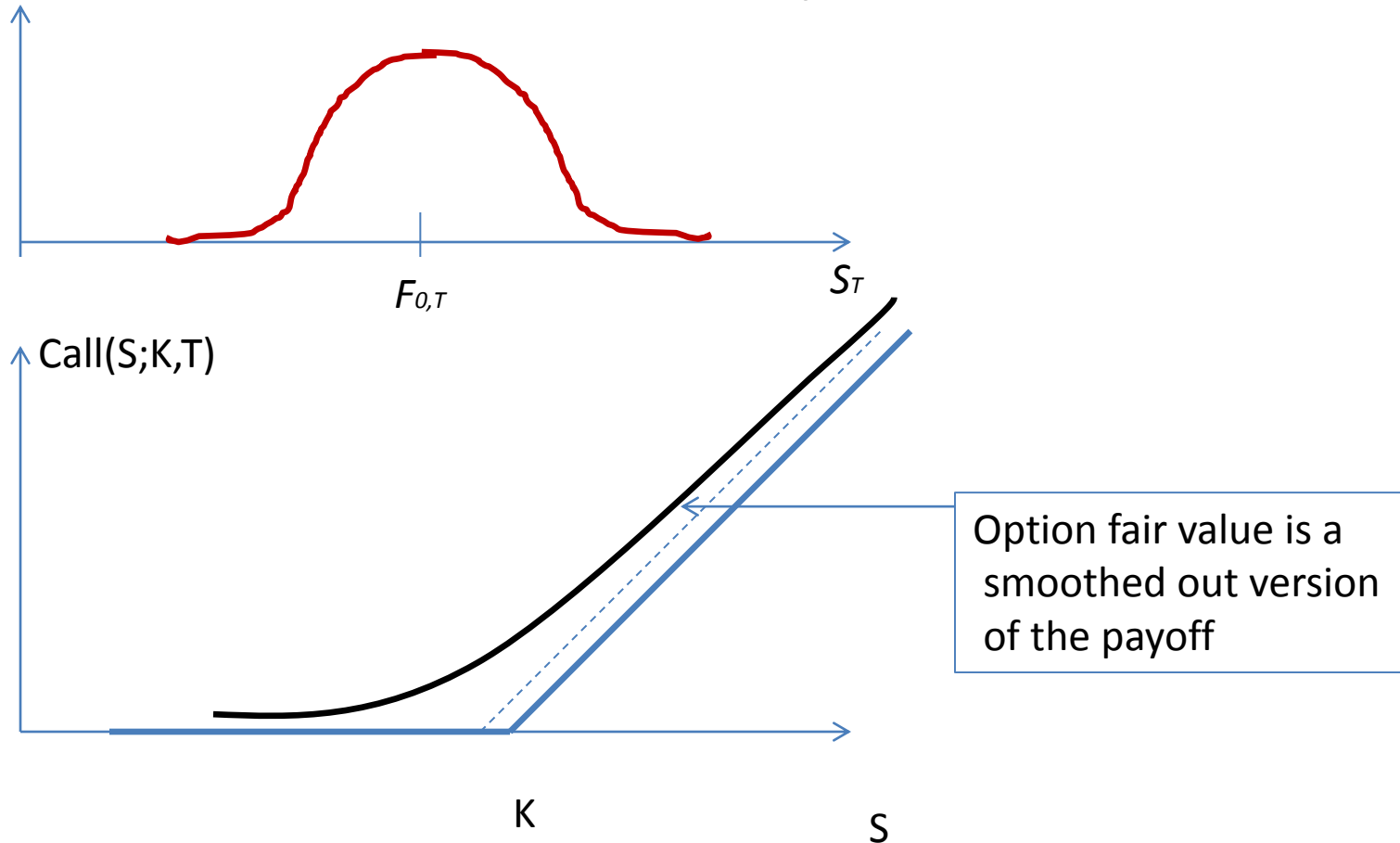
$$E^p \{S_T\} = F_{0,T}$$

- If we determine a suitable pricing measure, then all European options with expiration date T should have value given by

$$Call(K, T) = e^{-rT} E \left\{ (S_T - K)^+ \right\}, \quad Put(K, T) = e^{-rT} E \left\{ (K - S_T)^+ \right\}$$

- The main issue is then to determine a suitable pricing measure in the real, practical world.

What does a pricing measure achieve in the case of options?



The pricing measure gives a model to compute the option's fair value as a function of the price of the underlying asset, the strike and the maturity

The Black-Scholes Model

Assume that the pricing measure is log-normal, i.e. **log-returns are normal**

$$S_T = S_0 e^X, \quad X \sim N(\mu T, \sigma^2 T)$$

$$E\{S_T\} = \int_{-\infty}^{+\infty} S_0 e^y e^{-\frac{(y-\mu T)^2}{2\sigma^2 T}} \frac{dy}{\sqrt{2\pi\sigma^2 T}} = S_0 e^{\mu T + \frac{\sigma^2 T}{2}} = S_0 e^{\left(\mu + \frac{\sigma^2}{2}\right)T}$$

$$\therefore \quad \mu + \frac{\sigma^2}{2} = r - q \quad \therefore \quad \mu = r - q - \frac{\sigma^2}{2}$$

$$X = Z\sigma\sqrt{T} - \frac{\sigma^2}{2}T + (r - q)T, \quad Z \sim N(0,1)$$

Call pricing with the Black-Scholes model

$$\begin{aligned}
 \text{Call}(S, K, T) &= e^{-rT} E\left\{(S_T - K)^+\right\} = e^{-rT} \int_{-\infty}^{+\infty} \left(Se^{z\sigma\sqrt{T}-\sigma^2T/2+(r-q)T} - K\right)^+ e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \\
 &= e^{-rT} \int_A^{+\infty} Se^{z\sigma\sqrt{T}-\sigma^2T/2+(r-q)T} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} - e^{-rT} K \int_A^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \quad \left(A = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{K}{S}\right) + \frac{\sigma^2 T}{2} - (r-q)T \right) \right) \\
 &= e^{-qT} S \left(\int_A^{+\infty} e^{z\sigma\sqrt{T}-\sigma^2T/2} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left(\int_A^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) \\
 &= e^{-qT} S \left(\int_A^{+\infty} e^{-\frac{(z-\sigma\sqrt{T})^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left(\int_A^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) \\
 &= e^{-qT} S \left(\int_{A-\sigma\sqrt{T}}^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left(\int_A^{+\infty} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) \\
 &= e^{-qT} S \left(\int_{-\infty}^{-A+\sigma\sqrt{T}} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right) - e^{-rT} K \left(\int_{-\infty}^{-A} e^{-\frac{z^2}{2}} \frac{dz}{\sqrt{2\pi}} \right)
 \end{aligned}$$

Black-Scholes Formula

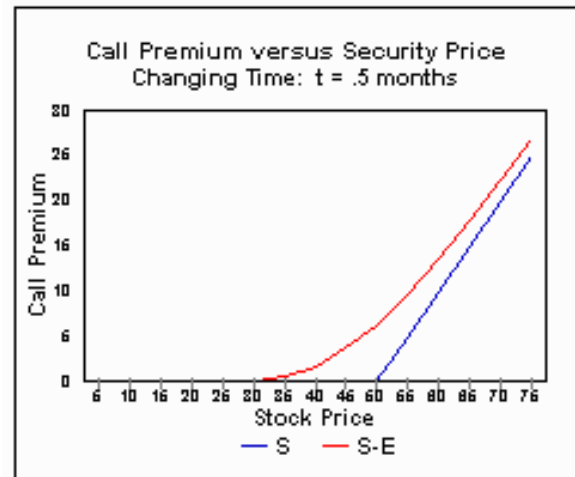
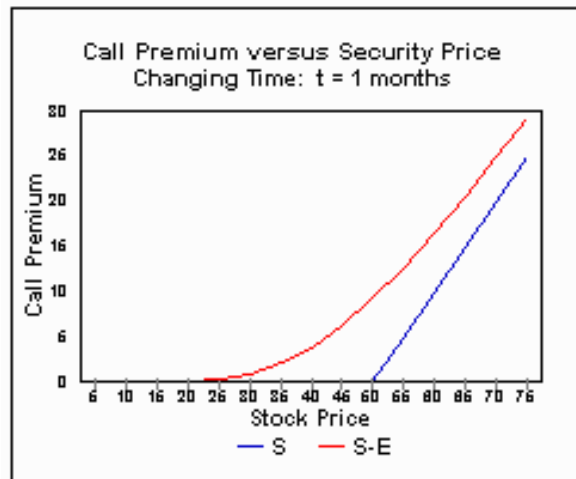
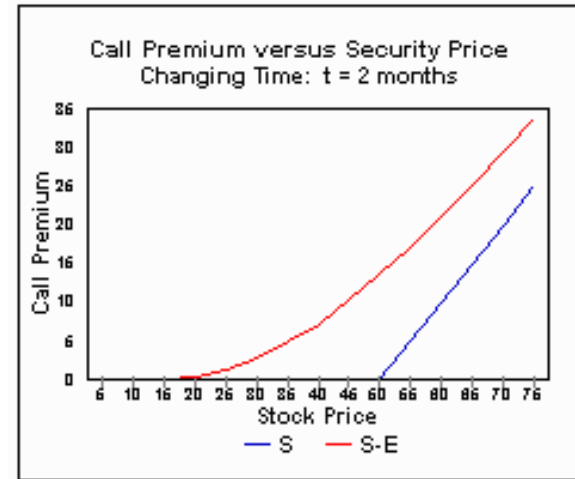
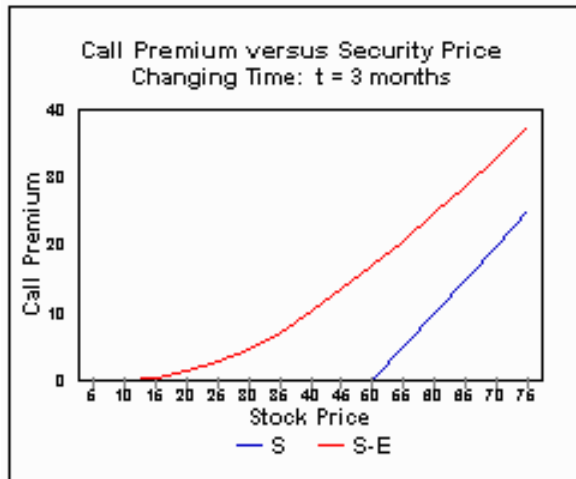
$$BSCall(S, T, K, r, q, \sigma) = Se^{-qT} N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{F_{0,T}}{K}\right) + \frac{\sigma^2 T}{2} \right), \quad d_2 = \frac{1}{\sigma\sqrt{T}} \left(\ln\left(\frac{F_{0,T}}{K}\right) - \frac{\sigma^2 T}{2} \right), \quad F_{0,T} = Se^{(r-q)T}$$

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{z^2}{2}} dz$$

cumulative normal distribution

Black-Scholes Formula at work



$S = \$48$, $K = \$50$, $r = 6\%$, $\sigma = 40\%$, $q = 0$

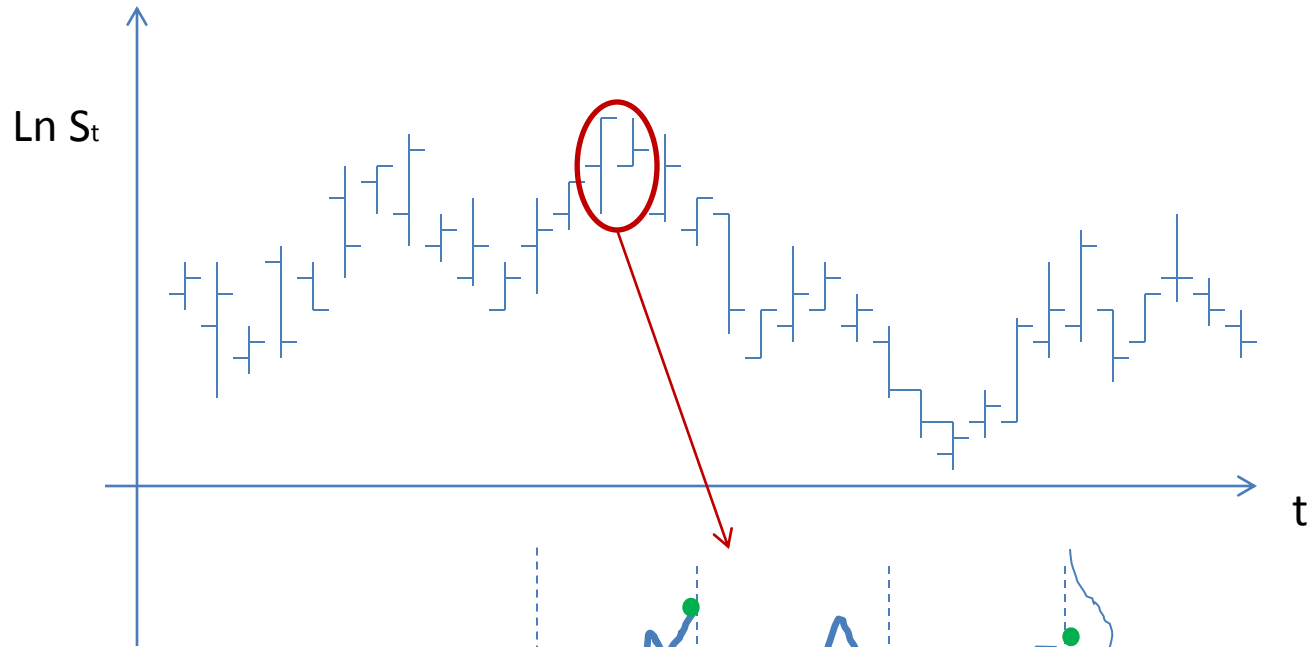
Multi-period asset model

- Derivative securities may depend on multiple expiration/cash-flow dates. Furthermore, the 1-period model described above is rigid in the sense that it cannot price American-style options.
- We consider instead a more realistic approach to pricing based on the statistics of stock returns over short periods of time (e.g. 1 day).
- We assume that the underlying price has returns satisfying

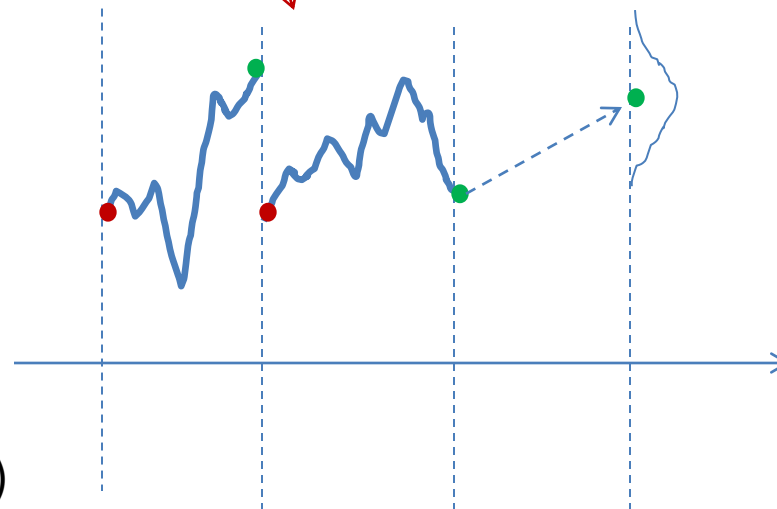
$$\frac{S_{t+\Delta t} - S_t}{S_t} = \frac{\Delta S_t}{S_t} \sim N(\mu\Delta t, \sigma^2\Delta t)$$

- We also assume that successive returns are uncorrelated.

Modeling the return of a price time-series (OHLC)



Model closing prices,
for example. The % changes
between closing prices
are normal and uncorrelated
 S_t = closing price of period $(t-1,t)$



Parameterization

$$\mu\Delta t = E\left\{\frac{\Delta S_t}{S_t}\right\}, \quad \sigma^2\Delta t = E\left\{\left(\frac{\Delta S_t}{S_t}\right)^2\right\} - \left(E\left\{\frac{\Delta S_t}{S_t}\right\}\right)^2$$

μ = annualized expected return

σ = annualized standard deviation

1% daily standard deviation => 15.9% annualized standard deviation

$$\Delta t = \frac{1}{252}, \quad \sqrt{252} = 15.9$$

Pricing Derivatives

- Let us model the value of a derivative security as a **function of the underlying asset price and the time to expiration**

$$V_t = C(S_t, t) \quad 0 < t < T$$

Change in market value over one period :

$$\Delta V_t = \Delta C(S_t, t)$$

$$= \frac{\partial C(S_t, t)}{\partial t} \Delta t + \frac{\partial C(S_t, t)}{\partial S} \Delta S + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial t^2} \Delta t^2 + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} (\Delta S_t)^2 + \dots$$

$$= \frac{\partial C(S_t, t)}{\partial t} \Delta t + \frac{\partial C(S_t, t)}{\partial S} \Delta S + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \left(\frac{\Delta S_t}{S_t} \right)^2 + o(\Delta t)$$

$$= \underbrace{\left(\frac{\partial C(S_t, t)}{\partial t} + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2 \right)}_{\alpha} \Delta t + \underbrace{\frac{\partial C(S_t, t)}{\partial S}}_{\beta} \Delta S_t + \underbrace{\frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \left[\left(\frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right]}_{\varepsilon_t} + o(\Delta t)$$

$$= \alpha \Delta t + \beta \Delta S_t + \varepsilon_t$$

The hedging argument

- Consider a portfolio which is long 1 derivative and short β stocks.
- Assume derivative does not pay dividends

$$\beta = \beta_t = \frac{\partial C(S_t, t)}{\partial S_t}$$

Profit and loss, including financing and dividends :

$$PNL = -V_t \cdot r\Delta t + \Delta V_t - \beta(\Delta S_t - S_t r\Delta t + S_t q\Delta t)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta\Delta S_t + \varepsilon_t - \beta(\Delta S_t - S_t r\Delta t + S_t q\Delta t)$$

$$= -V_t \cdot r\Delta t + \alpha\Delta t + \beta S_t (r - q)\Delta t + \varepsilon_t$$

$$= \left(-C(S_t, t)r + \frac{\partial C(S_t, t)}{\partial t} + \frac{\partial C(S_t, t)}{\partial S} S(r - q) + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2 \right) \Delta t + \varepsilon_t$$

Analyzing the residual term ε_t

$$\varepsilon_t = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} \left[\left(\frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \right] + o(\Delta t)$$

Conditional
expectation
of epsilon

$$E\{\varepsilon_t | S_t\} = S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} E\left\{ \left(\frac{\Delta S_t}{S_t} \right)^2 - \sigma^2 \Delta t \middle| S_t \right\} + o(\Delta t)$$

$$= S_t^2 \frac{\partial^2 C(S_t, t)}{\partial S^2} \mu^2 \Delta t^2 + o(\Delta t)$$

$$= o(\Delta t)$$

- The residual term has essentially zero expected return (vanishing exp. return in the limit $\Delta t \rightarrow 0$.)

The fair value of our derivative security is...

- The PNL for the long short portfolio of **1 derivative and – beta shares** has expected value

$$E\{PNL\} = \alpha\Delta t + o(\Delta t)$$
$$= \left(-C(S_t, t)r + \frac{\partial C(S_t, t)}{\partial t} + \frac{\partial C(S_t, t)}{\partial S} S(r - q) + \frac{1}{2} \frac{\partial^2 C(S_t, t)}{\partial S^2} S_t^2 \sigma^2 \right) \Delta t + o(\Delta t)$$

- This portfolio has no exposure to the stock price changes. Therefore, if $C(S_t, t)$ represents the “fair value” of the derivative, the portfolio should have zero rate of return (we already took into acct its financing). Thus:

$$\frac{\partial C(S, t)}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C(S, t)}{\partial S^2} + (r - q)S \frac{\partial C(S, t)}{\partial S} - C(S, t)r = 0$$

- This is the Black-Scholes partial differential equation (PDE).

American-style calls & puts

- Consider a call option on an underlying asset paying dividends continuously. Since the option can be exercised anytime, we have

$$C(S,t) \geq \max(S - K, 0), \quad t < T. \quad (1)$$

- The terminal condition at $t=T$ corresponds to the final payoff

$$C(S,T) = \max(S - K, 0).$$

- Thus, the function $C(S,t)$ should satisfy the **Black-Scholes PDE** in the region of the (S,t) -plane for which strict inequality holds in (1), and it should be equal to $\max(S-K,0)$ otherwise.
- The solution of this problem is done numerically and will be addressed in the next lecture.