# Derivative Securities: Lecture 4

Option Pricing: Arrow-Debreu approach

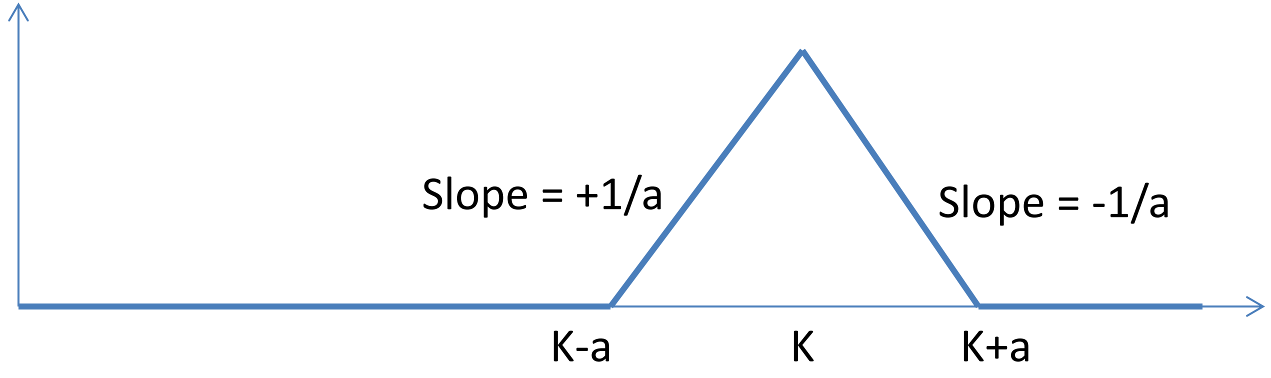
# A general approach to derivative pricing

- Assume a 1-period model: trades are done at date t=0 and the payoff date is t=T.
- Let $C(K,T)$ represent the fair value of a call option with strike K maturing on date $T$.
- Our first goal is to find a suitable pricing formula from first principles.
- Difficulty: we do not know the final state of the stock and the static cash and carry argument that we used for forwards does not apply. (Why?)

# Butterfly spreads

- A butterfly spread is a position in options which corresponds to

  - long 1 call with strike K-a
  - short 2 calls with strike K
  - long 1 call with strike K+a




Payoff diagram for butterfly spread

# Butterfly spreads and call prices

- Since the payoff of a butterfly is non-negative in all future states, its value should be positive

$$B(K-a,K,K+a,T) = C(K-a,T) - 2C(K,T) + C(K+a,T)$$

- Consider $1/a^2$ butterfly spreads in the limit $a \to 0$:

$$\lim_{a\to 0} \frac{B(K-a,K,K+a,T)}{a^2} = \lim_{a\to 0} \frac{C(K-a,T) - 2C(K,T) + C(K+a,T)}{a^2}$$

$$= \frac{\partial^2 C(K,T)}{\partial K^2} = B(K,T) \geq 0$$

- Conclusion: Call values should be convex in K (and decreasing).

# Arrow-Debreu prices

We have

$$\int_0^{\infty} B(X,T)dX = e^{-rT}$$

Proof: for K $\ll 1, C(K,T) \approx PV(F-K)$;

for K $\gg 1, C(K,T) \approx 0$

$$\frac{\partial C(K,T)}{\partial K} \to -e^{-rT} \; as \; K \to 0$$

$$\frac{\partial C(K,T)}{\partial K} \to 0 \; as \; K \to \infty$$

$$\int_0^{\infty} B(X,T)dX = \left[\frac{\partial C(K,T)}{\partial K}\right]_{K=0,\infty}$$

# AD Probabilities and Option Prices

$p(X,T) = e^{rT} B(K,T)$ Arrow-Debreu Probabilities

$$\int_0^{\infty} p(X,T)dX = 1$$

**Proposition 1:** Expected value under AD= forward price

$$\int_0^{\infty} X\, p(X,T)dX = F_T$$

**Proposition 2:** The fair values of puts and calls can be represented as their expectations under the AD probability

$$C(K,T)=e^{-rT} \int_0^{\infty} max(X-K,0)p(X,T)dX$$

$$P(K,T)=e^{-rT} \int_0^{\infty} max(K-X,0)p(X,T)dX$$

# Proof of Proposition 1

Proposition 1 follows from Proposition 2 because, from PCP,

$$C(K,T)\text{-}P(K,T) = e^{-rT}(F_T\text{-K}).$$

$$C(K,T) - P(K,T) = e^{-rT} \int_0^\infty max(X-K,0)p(X,T)dX -$$

$$e^{-rT} \int_0^\infty max(K-X,0)p(X,T)dX$$

$$= e^{-rT} \int_0^\infty (X-K)p(X,T)dX$$

$$= e^{-rT} \int_0^\infty Xp(X,T)dX - e^{-rT}K$$

$$\therefore \int_0^\infty Xp(X,T)dX = F_T$$

# Proof of Proposition 2

$$\frac{\partial}{\partial K}\ e^{-rT}\int_0^{\infty} max(X-K,0)p(X,T)dX =$$

$$= -\ e^{-rT}\int_0^{\infty} H(X-K)p(X,T)dX$$

$$= -\ e^{-rT}\int_K^{\infty} p(X,T)dX$$

$$\frac{\partial^2}{\partial K^2} = \ = \ e^{-rT}p(K,T) = B(K,T) = \frac{\partial^2 C(K,T)}{\partial K^2}$$

- Thus, the integral expression and the call price $C(K,T)$ differ at most by a linear function of $K$.

- It is trivial to show that since $C(0,T) = PV(F_T)$, $C(\infty,T) = 0$, the linear function is zero. The proof is the same for puts.

# Black Scholes model

- Assume that AD measure is log-normal. In other words, the AD probability is the distribution of a random variable

$$X = F_T e^{aZ+b}$$

where $Z$ is normal N(0,1) and a and b are parameters. Also we need from Prop 1, that

$$1 = E\left(e^{aZ+b}\right) = e^{\frac{a^2}{2}+b}$$

which implies $b = -\frac{a^2}{2}$, or $X = F_T e^{aZ-\frac{a^2}{2}}$.

- The parameter a, which is not a financial parameter (as the forward) corresponds to the standard deviation of log-returns, over the time horizon.

# Volatility

- The standard deviation of log returns is called volatility (option volatility, implied volatility)

- The units of volatility for equity derivatives and for FX (but not for all IR derivatives is *% change per year.*

- If we denote the volatility by $\sigma$ and measure time in years, we have

$$a = \sigma\sqrt{T}$$

and the BS model for the AD probabilities reads

$$X = F_T e^{\sigma\sqrt{T}Z - \frac{1}{2}\sigma^2 T}$$

- This means that $p(X, T)$ is the density of this random variable.

# The Black-Scholes Formula

Plugging in the density for the log-normal random variable, we find that

$$C(K,T)=e^{-rT}(F_T N(d_1) - KN(d_2))$$

where $N(x)$ is the cumulative distribution for a standardized normal r.v. and

$$d_1=\frac{1}{\sigma\sqrt{T}}\,ln\left(\frac{F_T}{K}\right)+\frac{\sigma^2 T}{2}$$

$$d_2=\frac{1}{\sigma\sqrt{T}}\,ln\left(\frac{F_T}{K}\right)-\frac{\sigma^2 T}{2}$$

This formula is used millions of times every day in exchanges worldwide

# The BS Formula with Spot quantities

- The BS formula also can be written in terms of ``spot'' quantities

$$\text{BSCall}(S,T,K,r,q,\sigma)=e^{-qT}SN(d_1)-e^{-rT}KN(d_2)$$

$$d_1=\frac{1}{\sigma\sqrt{T}}\,ln\left(\frac{S}{K}\right)+\frac{(r-q)\sqrt{T}}{\sigma}+\frac{\sigma^2T}{2}$$

$$d_2=\frac{1}{\sigma\sqrt{T}}\,ln\left(\frac{S}{K}\right)+\frac{(r-q)\sqrt{T}}{\sigma}-\frac{\sigma^2T}{2}$$

- It shows that the value of an option depends not only on the cost-of-carry but also on the volatility parameter.

- For historical reasons, the above formula is known as the Black-Scholes formula, whereas the one with the forward is known as Black's formula. Of course, they express exactly the same idea.