Conquering the Greeks in Monte Carlo: Efficient Calculation of the Market Sensitivities and Hedge-Ratios of Financial Assets by Direct Numerical Simulation

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Abstract

The calculation of price-sensitivities of contingent claims is formulated in the framework of Monte Carlo simulation. Rather than perturbing the parameters that drive the economic state-variables of the model, we perturb the vector of probabilities of simulated paths in a neighborhood of the uniform distribution. The resulting hedge-ratios (sensitivities with respect to input prices) are characterized in terms of higher-order moments of simulated cashflows. The computed sensitivities dislay excellent agreement with analytic closed-form solutions whenever the latter are available, *e.g.* with the Greeks of the Black-Scholes model, and with approximate analytic solutions for Basket Options in multi-asset models. The advantage of the new sensitivities is that they are "universal" (non-parametric) and simple to compute: they do not require performing multiple MC simulations, discrete-differentiation, or re-calibration of the simulation.

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1 Introduction

The goal of this note is to clarify recent proposals made for pricing and hedging derivative securities using Monte Carlo simulations with weighted paths (Avellaneda, Buff, Friedman, Grandchamp, Kruk and Newman (1999)). A typical Monte Carlo (MC) simulation assigns equal probability to each simulation path. In the weighted MC framework, paths can be assigned different probabilities. This added flexibility allows us to better fit model prices of benchmark instruments to the actual prices observed in the market. It also gives rise to a new method for computing price-sensitivities – which applies to both weighted and classical MC simulations. The question of interest here to analyze the sensitivities and the hedges that we compute by this new method and to compare them with standard approaches.

We begin by briefly reviewing the Weighted Monte Carlo approach. Fitting prices by assigning different probabilities to the simulated paths leads to the problem of how to choose the probabilities. Since the number of paths is typically much greater than the number of prices to be fitted, there exist many probability vectors which are consistent with a given set of observed market prices. It is natural to select the pricing probabilities by a penalization procedure which gives rise to a unique solution and makes the pricing scheme stable with respect to perturbations in the input prices.

To be specific, let N represent the number of paths and let M represent the number of benchmark instruments under consideration. Denote by $C_1, ..., C_M$ the spot market prices of these instruments. For each of the benchmark instruments, we consider the vector $\mathbf{g}^{(j)} = \left(g_1^{(j)}, \ .... \ g_N^{(j)}\right)$ where the entries represent the present value of the cash-flows of the instrument if the i^{th} path occurs ($i = 1, ... N$). If the path i has probability p_i, the M prices satisfy the equations

$$C_j = \sum_{i=1}^{N} g_i^{(j)} p_i \quad , \ j = 1, ... M \ . \tag{1}$$

In general, we have $M \ll N$ and this system of equation admits many solutions. A possible selection mechanism consists of choosing the probabilities p_i so as to minimize the quantity

$$\sum_{i=1}^{N} \Psi(p_i) \tag{2}$$

where $\Psi(x)$ is a convex function. The selection of a set of probabilities p_i which satisfy (1) and minimize (2) can be made by solving a Lagrange multipliers problem of the type

$$\min_{\lambda,\mu} \left\{ \sum_{i=1}^{N} \Psi(p_i) \; + \; \sum_{j=1}^{M} \lambda_j \left[\sum_{i=1}^{N} g_i^{(j)} \, p_i \; - \; C_j \right] \; + \; \mu \left[\sum_{i=1}^{N} p_i \; - \; 1 \right] \right\} .$$

The formal solution of the optimization problem is given by

$$p_i \;=\; (\Psi')^{-1} \left[-\sum_{j=1}^{M} \lambda_j \, g_i^{(j)} \; - \mu \right]$$

or, equivalently,

$$p_i \;=\; \Psi^{*'} \left[-\sum_{j=1}^{M} \lambda_j \, g_i^{(j)} \; - \mu \right] . \tag{3}$$

Here, $\Psi^*(x)$ is the convex conjugate of $\Psi(x)$, in the sense of Legendre. The latter equation can be interpreted as defining an M-parameter family of distributions (corresponding to the probability distributions obtained by varying the vector of Lagrange multipliers $(\lambda_1, \, ...\lambda_M)$). The parameter μ is determined uniquely from the fact that the sum of the p_i's is one.

There exist three special cases of the Ψ-function that are noteworthy. First,

$$\Psi(x) \;=\; x \, ln(x)$$

which corresponds to minimization of the relative (Kullback-Leibler) entropy under constraints. Secondly, the function

$$\Psi(x) \;=\; \sqrt{x}$$

which is corresponds to the Skorohod distance between p and the uniform measure on paths. The third case corresponds to the quadratic function

$$\Psi(x) \;=\; (x - \frac{1}{N})^2 \; .$$

Albeit simple to compute, the quadratic penalization function may lead to negative probabilities – unlike the Kullback and Skorohod distances. It should therefore be used with caution.

We shall focus mostly on the relative entropy distance. The parametric family of probabilities which arises in this case is the well-known *Boltzman* distribution

$$p_i(\lambda) \;=\; \frac{exp\left(\sum\limits_{j=1}^{M} \lambda_j\, g_i^{(j)}\right)}{\sum\limits_{i=1}^{N}\; exp\left(\sum\limits_{j=1}^{M} \lambda_j\, g_i^{(j)}\right)} \;=\; \frac{1}{Z(\lambda)}\, exp\left(\sum_{j=1}^{M} \lambda_j\, g_i^{(j)}\right) \tag{4}$$

where

$$Z(\lambda) \;=\; \sum_{i=1}^{N}\; exp\left(\sum_{j=1}^{M} \lambda_j\, g_i^{(j)}\right) \tag{5}$$

is the partition function. A useful identity for finding the Lambdas numerically is

$$C_j \;=\; \frac{\partial ln Z(\lambda)}{\partial \lambda_j}\;, \tag{6}$$

which follows from substituting the expression for the p_i's in the pricing identity (1). This means that the values of the Lagrange multipliers that produce the correct probabilities in the entropic case are obtained by minimizing the *objective function*

$$ln(Z(\lambda)) - \sum_{j=1}^{M} \lambda_j\, C_j\;.$$

The minimization can be implemented numerically with a gradient-based optimization Quasi-Newton algorithm such as BFGS, or with the classical Newton algorithm that utilizes both the gradient and the Hessian of $ln(Z(\lambda))$. The gradient-based method uses $1 + M$ function evaluations, each of which has complexity $O(N)$. The full Newton approach requires instead $1 + M + \frac{M(M+1)}{2}$ function evaluations. So far, we have only tested the gradient-based method and obtained satisfactory results with up to 50-100 instruments in some cases. This may have to do with the fact that BFGS is a highly efficient algorithm which combines line-searches with quasi-Newton steps using a pseudo-Hessian. The quasi-Newton method seems well-fitted to the objective functions that arise in the entropy optimization problem, which, despite being convex, are quite "flat" in most regions of space. In deciding whether to choose the quasi-Newton or the "pure" Newton approach, the user must therefore weigh the benefits of using an explicit Hessian against having a more costly evaluation step in the search. [1]

2 Computing sensitivities with respect to the input prices

We now come to the main subject of this paper, which is the calculation of price-sensitivities. One of the main conclusions of the section will be to show that the sensitivities are closely related to regression coefficients and that this result is, in some sense, independent of the penalization function Ψ.

Let $h = (h_1, \ldots h_N)$ represent the payoff vector of a portfolio of contingent claims. The model value of the portfolio is

$$\begin{aligned} E(h) &= \sum_{i=1}^{N} p_i h_i \\ &= \sum_{i=1}^{N} \Psi_i^{*'} h_i \end{aligned} \tag{7}$$

where we set $\Psi_i^{*'} = \Psi^{*'} \left[-\lambda \cdot g_i - \mu \right]$.

The problem of computing the sensitivities of such a portfolio can be cast formally as the calculation of the partial derivatives

$$\frac{\partial E(h)}{\partial C_j} , \quad j = 1, \ldots M.$$

If we consider the case of probability measures obtained by a penalization selection with penalty function $\Psi(x)$, as described above, we find that

[1]The case for computing with the full Newton algorithm has been made to me by Paul Fackler (private communication, April 2000). Also, Raphael Douady (1999) proposes a modified Newton algorithm that used the Hessian matrix to compute the minimum entropuy solution.

$$\begin{aligned}\frac{\partial E(h)}{\partial \lambda_j} &= \sum_{i=1}^{N} \frac{\partial p}{\partial \lambda_j} h_i \\ &= -\sum_{i=1}^{N} \left[\Psi_i^{*''} g_i^{(j)} + \Psi_i^{*''} \frac{\partial \mu}{\partial \lambda_j} \right] h_i \\ &= -\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)} h_i - \frac{\partial \mu}{\partial \lambda_j} \sum_{i=1}^{N} \Psi_i^{*''} h_i \end{aligned} \tag{8}$$

where we used the fact that $p_i = \Psi^{*'} [-\lambda \cdot g_i - \mu]$ and the abbreviation $\Psi_i^{*''} = \Psi^{*''} [-\lambda \cdot g_i - \mu]$. To obtain an "explicit" expression for $\frac{\partial \mu}{\partial \lambda_j}$, we differentiate equation (3) with respect to λ_j. The result is

$$\sum_{i=1}^{N} \left[\Psi_i^{*''} g_i^{(j)} + \Psi_i^{*''} \frac{\partial \mu}{\partial \lambda_j} \right] = 0,$$

which implies that

$$\frac{\partial \mu}{\partial \lambda_j} = -\frac{\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)}}{\sum_{i=1}^{N} \Psi_i^{*''}}.$$

Substituting back into (8), we find that

$$\begin{aligned}\frac{\partial E(h)}{\partial \lambda_j} &= -\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)} + \frac{\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)}}{\sum_{i=1}^{N} \Psi_i^{*''}} \sum_{i=1}^{N} \Psi_i^{*''} h_i \\ &= -\left(\sum_{i=1}^{N} \Psi_i^{*''} \right) \cdot \left(\frac{\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)} h_i}{\sum_{i=1}^{N} \Psi_i^{*''}} - \frac{\sum_{i=1}^{N} \Psi_i^{*''} g_i^{(j)}}{\sum_{i=1}^{N} \Psi_i^{*''}} \cdot \frac{\sum_{i=1}^{N} \Psi_i^{*''} h_i}{\sum_{i=1}^{N} \Psi_i^{*''}} \right) \end{aligned} \tag{9}$$

Notice that the expression in parenthesis can be interpreted as a covariance. More precisely, if we define the new expectation operator (and its associated probability) by the equation

$$E^* (f) = \frac{\sum_{i=1}^{N} \Psi_i^{*''} f_i}{\sum_{i=1}^{N} \Psi_i^{*''}},$$

we can write, more concisely,

$$\frac{\partial E(h)}{\partial \lambda_j} = -\left(\sum_{i=1}^{N} \Psi_i^{*''}\right) \cdot Cov^*\left(g^{(j)}, h\right).$$

Clearly, the same argument can be applied to the case $h = g^{(k)}$, so we also have

$$\frac{\partial E(g^{(k)})}{\partial \lambda_j} = -\left(\sum_{i=1}^{N} \Psi_i^{*''}\right) \cdot Cov^*\left(g^{(j)}, g^{(k)}\right).$$

Finally, since we have $C_k = E(g^{(k)})$, we derive an expression for the sensitivities with respect to the *input prices*, namely,

$$\frac{\partial E(h)}{\partial C_k} = \sum_{j=1}^{M} \frac{\partial E(h)}{\partial \lambda_j} \cdot \frac{\partial \lambda_j}{\partial C_k}$$

or, in matrix notation,

$$\nabla_C E(h) = Cov^*\left(\mathbf{g}, h\right) \cdot \left(Cov^*\left(\mathbf{g}, \mathbf{g}\right)\right)^{-1}. \tag{10}$$

This equation implies that

Proposition 1 *: The vector of price-sensitivities obtained with the method perturbation of measures with penalization function $\Psi(x)$ is equal to the vector of regression coefficients (under the measure induced by $E^*(\bullet)$) of the payoff of the target portfolio on the linear space generated by the cash-flow vectors of the input instruments $\left(\mathbf{g}^{(1)}, ...\mathbf{g}^{(M)}\right)$. More precisely, we have*

$$\frac{\partial E(h)}{\partial C_k} = \beta_k$$

where

$$\begin{aligned} \boldsymbol{\beta} &= \arg\min_{\beta} E^*\left(h - \sum_{j=1}^{M} \beta_j \mathbf{g}^{(j)}\right)^2 \\ &= \arg\min_{\beta} \frac{E\left[\left(\frac{\Psi^{*\prime\prime}}{\Psi^{*\prime}}\right)\left(h - \sum_{j=1}^{M} \beta_j \mathbf{g}^{(j)}\right)^2\right]}{E\left[\left(\frac{\Psi^{*\prime\prime}}{\Psi^{*\prime}}\right)\right]}. \end{aligned} \tag{11}$$

It is important to note at this point the special role played by the Kullback-Leibler entropy distance. Since in this case we have $\Psi(x) = x\ln(x)$, and hence

$$\Psi^*(x) = e^{x-1} = \Psi^{*'}(x) = \Psi^{*''}(x),$$

we have

$$\Psi_i^{*''} = \Psi_i^{*'} = p_i.$$

Therefore,

Proposition 2 *If the penalization corresponds to the Kullback-Leibler relative entropy distance, the hedge-ratios are equal to the regression coefficients of the cashflow vector of the portfolio onto the cash-flow vectors of the benchmark instruments under the pricing measure.*

This is a classical result, derived previously in Avellaneda (1998) and in Avellaneda, Buff, Friedman, Grandchamp, Kruk and Newman (1999).

In situations of practical interest, the prior probability measure may be such that the simulation is already calibrated to the observed prices of benchmark instruments. This would arise, for instance, if the underlying model parameters were fitted using a classical procedure, such as least-squares fitting or another optimization procedure. Under these circumnstances, the method of Weighted Monte Carlo can still be used to compute sensitivities and hedge ratios and gives non-trivial results. In fact, assume that the model is calibrated to the prices of benchmark instruments with p =the uniform measure on paths. If this is the case, we have $\Psi_i^{*''} = const.$ and $\Psi_i^{*'} = \frac{1}{N}$. In particular, we have $E^*(\bullet) = E(\bullet) =$ expectation over the uniform measure, and thus the

Corollary 3 *If the Monte Carlo simulation is calibrated with $\lambda = 0$, the hedge-ratios produced by all $\Psi-$penalizations are equal. They are equal to the coefficients of the least-squares projection of the vector of cash-flows of the target portfolio on the space generated by the cash-flows of the benchmark instruments.*

3 The parametric approach

It is natural to compare the above results with "parametric approach" to sensitivity-analysis, which consists iof embedding the risk-neutral measure in an M-parameter family of risk-neutral measures and performing a perturbation analysis. Consider therefore a parametric, but "non-entropic", family of probability vectors $p_i(\theta), i = 1, ..., N, \theta \in R^M$.

The pricing equations are now

$$E(h) = \sum_{i=1}^{N} p_i(\theta) h_i$$

and

$$E(g^{(j)}) = \sum_{i=1}^{N} p_i(\theta) g_i^{(j)},$$

for $j = 1, ..., M$. (For simplicity, we assume that the number of parameters is the same as the number of input instruments). Differentiation of these equations with respect to θ gives

$$\begin{aligned} \nabla_\theta E(h) &= \sum_{i=1}^{N} \nabla_\theta p_i(\theta) h_i \\ &= E\left(\frac{\nabla_\theta p(\theta)}{p(\theta)} h\right) \\ &= Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)}, h\right) \end{aligned}$$

and

$$\begin{aligned} \nabla_\theta E(g^{(j)}) &= \sum_{i=1}^{N} \nabla_\theta p_i(\theta) g_i^{(j)} \\ &= E\left(\frac{\nabla_\theta p(\theta)}{p(\theta)} g^{(j)}\right) \\ &= Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)}, g^{(j)}\right). \end{aligned}$$

(Notice that we used here the identity

$$E\left(\frac{\nabla_{\theta}p(\theta)}{p(\theta)}\right)=\mathbf{0},$$

which holds because $p(\theta)$ is a probability.) We conclude that the "parametric" expression for hedge-ratios takes the form

$$\nabla_{C}E(h)=Cov\left(\frac{\nabla_{\theta}p(\theta)}{p(\theta)},h\right)\cdot\left[Cov\left(\frac{\nabla_{\theta}p(\theta)}{p(\theta)},\mathbf{g}\right)\right]^{-1}. \qquad (12)$$

Thus, as in the "non-parametric approach" the hedge-sensitivities with respect to prices are expressed in terms of covariances involving the different vectors of cash-flows. The main difference is that the parametric form of the distribution is manifested with the appearance of the gradient of the "likelihood function"

$$\nabla_{\theta}\ln p(\theta)=\frac{\nabla_{\theta}p(\theta)}{p(\theta)}.$$

The reader will note that these formulas can be used, in conjunction with the explicit expressions for $p(\theta)=p(\lambda)$ of the non-parametric case to rederive the results of the previous sections. The quantity that corresponds to the likelihood function in the case of a $\Psi-$perturbation is simply the ratio

$$\frac{\Psi_i^{*\prime\prime}}{\Psi_i^{*\prime}}.$$

Formulas for computing Greek sensitivities for option prices along these lines were first proposed by Broadie and Glasserman (1996).

This "parametric approach" is based on differentiation of the probability measure on path space with respect to the parameter vector θ. With the exception of simple cases Broadie and Glasserman (1996), the dependence between the parameters and the probabilities induced by the corresponding paths is not explicit and requires the use of Calculus of Variations in path-space, or Malliavin Calculus (see Fournie, Lasry, Lebuchoux, Lions and Touzi (1998)). Unfortunately, a "direct" Malliavin calculus approach – i.e. the calculation of the *exact* derivatives of the expectation with respect to the parameters θ – will be difficult if not impossible to implement in practice due to the complexity of the typicaly pricing models. This is more so as we require that the probability measure be fitted to the price of many benchmark instruments (requiring the fitting of may parameters in the calibration step). In contrast, the approach that we advocate here requires only performing regressions of different cash-flow vectors – without carrying out any explicit differentiation with respect to the "internal" model parameters..

A theoretical comparison between the "parametric hedge-ratios" which arise by varying the parameter θ and the non-parametric hedge-ratios obtained by regressions is developed in the Appendix. There, we show that the WMC method is consitent with (i.e. converges to) the true sensitivities as the number of benchmarks increases (in a suitable sense). In practice, of course, we obtain surprisingly good results with a relatively small set of benchmark instruments.

In the following section, we develop explicit formulas for Delta and Gamma using the technique of Weighted Monte Carlo.

4 Computation of the Classical "Greeks"in MC simulations

The weighted MC method produces the sensitivities with respect to *input instruments* by computing a least-squares regression of cash-flows. On the other hand, traders might be interested in sensitivities with respect to *spot prices*, such as Deltas and Gammas. The method presented here can be used to calculate these "Greek" sensitivities, *provided that the sensitivities of the benchmark prices to the parameters of interest can be computed in closed-form.*

To fix ideas, we shall consider a Black-Scholes world in which the input instruments are either options or forwards. We assume that the prices of the benchmark instruments are now given by closed-form (Black-Scholes) expressions

$$C_j = C_j\left(S\right)$$

where S is the spot price. The Delta of the contingent claim with payoff h is

$$\frac{\partial E\left(h\right)}{\partial S} = \sum_{j=1}^{M} \frac{\partial E\left(h\right)}{\partial C_j}\frac{\partial C_j}{\partial S}$$

$$
\begin{aligned}
&= \nabla_C E(h) \cdot \nabla_S \mathbf{C} \\
&= \boldsymbol{\beta} \cdot \nabla_S \mathbf{C} \qquad (13)
\end{aligned}
$$

Thus, the total exposure to a movement in the spot price is computed by converting the exposures to different instruments into "Delta-equivalents", using the Black-Scholes formula to compute the Delta of each benchmark.

The calculation of Gamma is more complicated, but follows the same approach:

$$
\begin{aligned}
\frac{\partial^2 E(h)}{\partial S^2} &= \frac{\partial}{\partial S}\left(\sum_{j=1}^{M} \beta_j \frac{\partial C_j}{\partial S}\right) \\
&= \sum_{j=1}^{M} \frac{\partial \beta_j}{\partial S}\frac{\partial C_j}{\partial S} + \sum_{j=1}^{M} \beta_j \frac{\partial^2 C_j}{\partial S^2}. \qquad (14)
\end{aligned}
$$

The key point is the computation of the quantities $\frac{\partial \beta_j}{\partial S}$. For this, we proceed as follows

$$
\begin{aligned}
\frac{\partial \beta_j}{\partial S} &= \sum_{k=1}^{M} \frac{\partial \beta_j}{\partial C_k}\frac{\partial C_k}{\partial S} \\
&= \sum_{k=1}^{M}\sum_{l=1}^{M} \frac{\partial \beta_j}{\partial \lambda_l}\frac{\partial \lambda_l}{\partial C_k}\frac{\partial C_k}{\partial S}. \qquad (15)
\end{aligned}
$$

To continue, we use the correspondence between $\lambda-$derivatives and covariances, which holds for the Kullback-Leibler perturbations. Accordingly, let

$$Q = (Cov(\mathbf{g}, \mathbf{g}))^{-1}.$$

We have

$$\frac{\partial Q}{\partial \lambda_l} = -Q \cdot \left(\frac{\partial}{\partial \lambda_l} Cov(\mathbf{g}, \mathbf{g})\right) \cdot Q$$

and hence

$$
\begin{aligned}
\frac{\partial \boldsymbol{\beta}}{\partial \lambda_l} &= \frac{\partial Q}{\partial \lambda_l} \cdot Cov(h, \mathbf{g}) + Q \cdot \left(\frac{\partial}{\partial \lambda_l} Cov(h, \mathbf{g})\right) \\
&= -Q \cdot \left(\frac{\partial}{\partial \lambda_l} Cov(\mathbf{g}, \mathbf{g})\right) \cdot Q \cdot Cov(h, \mathbf{g}) + Q \cdot \left(\frac{\partial}{\partial \lambda_l} Cov(h, \mathbf{g})\right) \qquad (16)
\end{aligned}
$$

The derivatives inside this expression can be computed explicitly using the identity

$$\frac{\partial E(f)}{\partial \lambda_l} = Cov\left(f, g^{(l)}\right)$$

which is valid for all f. The final result, after substituting the expressions for $\frac{\partial \beta_j}{\partial \lambda_l}$ into the expression for Gamma, is

$$\frac{\partial^2 E(h)}{\partial S^2} = \sum_{j=1}^{M} \beta_j \frac{\partial^2 C_j}{\partial S^2} - \sum_{j,k=1}^{M} \frac{\partial C_j}{\partial S} M_{jk} \frac{\partial C_k}{\partial S}, \tag{17}$$

where

$$M_{jk} = \sum_{p,q=1}^{M} Q_{jp} N_{pq} Q_{qk},$$

with

$$\begin{aligned} N_{pq} &= E\left(\gamma^{(p)} \gamma^{(q)} \eta\right), \\ \gamma_i^{(p)} &= g_i^{(p)} - C_p, \\ \gamma_i^{(q)} &= g_i^{(q)} - C_q, \end{aligned}$$

and

$$\eta_i = h_i - \boldsymbol{\beta} \cdot \mathbf{g}_i - \left(E(h) - \boldsymbol{\beta} \cdot \mathbf{C}\right).$$

These expressions can be easily evaluated numerically.

5 Numerical results

We conducted several numerical experiments in the case of MC simulations based on lognormal pricing models. In the first experiment, we considered a single-asset Black-Scholes model with the following inputs

$$\begin{aligned} S &= \$146 \\ \sigma &= 58\% \\ r &= 5\% \\ d &= 0\% \end{aligned}$$

The day count basis was 365 days/yr. The benchmark instruments chosen were:

Type	Expiration(days)	Strike	Mkt. Price
fwd	135	-	-
call	135	143	22.98
fwd	115	-	-
call	115	143	21.27
fwd	105	-	-
call	105	143	20.37
fwd	95	-	-
call	95	143	19.42
fwd	85	-	-
call	85	143	18.42
fwd	75	-	-
call	75	143	17.37
fwd	65	-	-
call	65	143	16.25
fwd	55		-
call	55	143	15.04
fwd	45		-
call	45	143	13.73
fwd	35	-	-
call	35	143	12.26
fwd	25	-	-
call	25	143	10.58
fwd	15		-
call	15	143	8.54
fwd	5		-
call	5	143	5.65
fwd	125	-	-
call	125	143	22.14
fwd	90	-	-
fwd	60	-	-

These prices correspond exactly to the Black-Scholes option prices with the above parameters, so that the model is assumed to be nearly calibrated with $\lambda = 0$. We simulated 7000 paths in each simulation and performed 10 simulations for each pricing event to eliminate the dependence of the seed. Each pricing event was calibrated exactly using the Kullback entropy method. The average magnitude for the entropy distance was on the order of 10^{-3}. For each pricing event, we computed the prices, deltas and gammas of the different options. We carried out two types of simulations:ones which involved minimum-entropy calibration (and thus $\lambda \neq 0$) and others in which we did not calibrate (thus $\lambda = 0$). In the latter cases, we still used formulas (13) and (17) to compute the Deltas and Gammas, respectively.

Example 1: European put, expiration: 90 days, strike $135.

-	Price	Delta	Gamma
BS	10.4462	-0.323301	0.00853
Weighted MC	10.4519	-0.320309	0.00874
MC ($\lambda = 0$)	10.4395	-0.320268	0.00873

Example 2: European call, expiration 60 days, strike $190.

-	Price	Delta	Gamma
BS	2.74417	0.166784	0.00728
Weighted MC	2.72814	0.162294	0.00711
MC ($\lambda = 0$)	2.70869	0.161862	0.00708

Example 3: European call, expiration 60 days, strike $145.

-	Price	Delta	Gamma
BS	14.6858	0.57118	0.011377
Weighted MC	14.6759	0.57204	0.011677
MC ($\lambda = 0$)	14.6322	0.57195	0.011668

Additional experiments on European-style options on a single asset seem to indicate that the WMC procedure and the formulas for the Greeks gives accurat deltas and gammas for a wide range of strikes and maturities. The robustness of the method is quite good in the sense that the sensitivities are computed with high accuracy even for options which are deeply out of the money (e.g.15% deltas).

We also experimented with pricing Basket Options on equities. We considered the following parameters:

-	Stock 1	Stock 2	Stock 3
Spot Price	86	340	73.25
Vol	60	60	35

and correlation matrix

-	Stock 1	Stock 2	Stock 3
Stock 1	100	43.42	24.76
Stock 2	43.42	100	20.19
Stock 3	24.76	20.19	100

The benchmark securities used in the calibration were

Stock	Type	Expiration	Strike	Price
1	fwd	45	-	-
1	call	45	85	9.36
1	fwd	40	-	-
1	fwd	35	-	-
1	call	35	80	10.98
1	fwd	32	-	-
1	fwd	30	-	-
1	call	30	90	5.54
1	fwd	25	-	-
1	fwd	15	-	-
2	fwd	45	-	-
2	call	45	341	34.73
2	fwd	40	-	-
2	fwd	35	-	-
2	fwd	25	-	-
2	fwd	15	-	-
3	fwd	45	-	-
3	call	45	73	4.66
3	fwd	40	-	-
3	fwd	35	-	-
3	call	35	75	3.18
3	fwd	25	-	-
3	fwd	15	-	-

The interest rate was taken to be 4% and stock dividends were neglected. We considered a few examples and compared the corresponding results to the well-known Black-Scholes approximation for the value and Greeks of basket options, which is commonly used by equity derivatives traders.

The Black-Scholes approximation for the price of the basket is computed by inputting a constant volatility of

$$\sigma_{basket}^2 = \frac{\sum_i n_i^2 S_i^2 \sigma_i^2 + \sum_{i \neq j} n_i n_j S_i S_j \sigma_i \sigma_j}{\left(\sum n_i S_i\right)^2}.$$

The "basket delta" for MC was calculated by the formula:

$$\Delta_{basket} = \frac{1}{3}\left(\frac{\delta_1}{n_1} + \frac{\delta_2}{n_2} + \frac{\delta_3}{n_3}\right),$$

where δ_i represent the deltas with respect to each stock in the multi-asset model and n_i represents the amounts of shares of each stock in the basket. In the case of BS, individual Deltas are obtained by multiplying the Basket Delta (given in closed form) by the number or shares of each stock held. With the data used for this example, we have $\sigma_{basket} = 38.69\%$. We used a day-count convemtion of 255 days/yr., simulations with 5000 paths, and priced each option 10 times. The results represent the averages of the 10 pricing events.

Example 4. Put option on a basket composed of \$100 in each of the three stock at inception. Expiration: 40 days, strike price: \$300.

-	Price	Delta(1)	Delta(2)	Delta(3)	Basket Delta
BS	17.3301	-0.52720	-0.13335	-0.61896	-0.45410
WMC	17.3672	-0.51555	-0.12877	-0.66648	-0.45339
MC($\lambda = 0$)	17.3083	-0.51548	-0.12875	-0.66656	-0.45644

We also computed de Gamma matrix:

Γ	Stock 1	Stock 2	Stock 3
Stock 1	0.0113		
Stock 2	0.0026	0.0008	
Stock 3	0.0156	0.0034	0.0154

Example 5. Call option on a basket composed of \$100 in each of the three stock at inception. Expiration: 30 days, strike price: \$350.

-	Price	Delta(1)	Delta(2)	Delta(3)	Basket Delta
BS	2.79104	0.16853	0.04263	0.19787	0.14494
WMC	2.84153	0.16861	0.04747	0.17639	0.13107
MC($\lambda = 0$)	2.79141	0.16862	0.04748	0.17502	0.13037

A more detailed study of the sensitivities for basket options, including Vega-sensitivities will appear in a separate article (Avellaneda and Gamba, forthcoming).

6 Appendix: Comparison between the true sensitivities and the ones obtained by regression.

In this section, we assume that the risk-neutral measure is given by a parametric family of distributions $p_i(\theta)$, where θ is a parameter. We also assumed that the model is calibrated to match the prices of M reference instruments. The sensitivities computed by the standard method can be obtained by differentiation with respect to θ and applying the chain rule to obtain sensitivities with respect to prices. The sensitivities obtained by the "entropy method" or, more generally, by $\Psi-$penalizations, correspond to the "betas" of linear regressions of cash-flows. In this Appendix, we show how the two concepts are mathematically related.

Let $\boldsymbol{\beta}_p$ and $\boldsymbol{\beta}_{np}$ denote the vectors of sensitivities obtained by the parametric and non-parametric methods, respectively. Recall that

$$\boldsymbol{\beta}_p = \left(Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)}, \mathbf{g} \right) \right)^{-1} \cdot Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)}, h \right)$$

and

$$\boldsymbol{\beta}_{np} = \left(Cov\left(\mathbf{g}, \mathbf{g} \right) \right)^{-1} \cdot Cov\left(\mathbf{g}, h \right).$$

Treating $\frac{\nabla_\theta p(\theta)}{p(\theta)}$ as a random vector, we consider a linear regression of this vector on the space generated by the cash-flow vectors $g^{(j)} - E\left(g^{(j)}\right), j =$

$1,...,M$. Accordingly, we have

$$\frac{1}{p(\theta)}\frac{\partial p(\theta)}{\partial\theta_j}=\sum_{k=1}^{M}a_{jk}\left(g^{(j)}-E\left(g^{(j)}\right)\right)+\epsilon_j$$

where $a_{jk}=a_{jk}(\theta)$ are regression coefficients and ϵ_j are random variables (residuals) with mean zero. (Notice that we use here the fact that $\frac{\nabla_\theta p(\theta)}{p(\theta)}$ has mean zero.) We can rewrite the above equation in matrix-vector notation as

$$\frac{\nabla_\theta p(\theta)}{p(\theta)}=A(\theta)\cdot(\mathbf{g}-E(\mathbf{g}))+\boldsymbol{\epsilon} \tag{18}$$

In particular, it follows that

$$Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)},\mathbf{g}\right)=A(\theta)\cdot Cov(\mathbf{g},\mathbf{g})$$

Similarly, we have

$$Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)},h\right)=A(\theta)\cdot Cov(\mathbf{g},h)+Cov(\boldsymbol{\epsilon},h).$$

Hence,

$$\begin{aligned}\boldsymbol{\beta}_p &= \left(Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)},\mathbf{g}\right)\right)^{-1}\cdot Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)},h\right)\\ &= (Cov(\mathbf{g},\mathbf{g}))^{-1}\cdot(A(\theta))^{-1}\left[A(\theta)\cdot Cov(\mathbf{g},h)+Cov(\boldsymbol{\epsilon},h)\right]\\ &= \boldsymbol{\beta}_{np}+(A(\theta))^{-1}\cdot Cov(\boldsymbol{\epsilon},h)\\ &= \boldsymbol{\beta}_{np}+(Cov(\mathbf{g},\mathbf{g}))^{-1}\cdot(A(\theta))^{-1}\cdot Cov\left(\boldsymbol{\epsilon},h-\boldsymbol{\beta}_{np}\cdot\mathbf{g}\right).\end{aligned}$$

The last expression was obtained using the fact that the residual of $\frac{\nabla_\theta p(\theta)}{p(\theta)}$ is uncorrelated with $\mathbf{g}$. Thus, formally, we have

Proposition 4 *: The difference between the parametric and non-parametric hedge-ratios satisfies:*

$$\begin{aligned}\boldsymbol{\beta}_p-\boldsymbol{\beta}_{np} &= (Cov(\mathbf{g},\mathbf{g}))^{-1}\cdot(A(\theta))^{-1}\cdot Cov\left(\boldsymbol{\epsilon},h-\boldsymbol{\beta}_{np}\cdot\mathbf{g}\right)\\ &= (Cov(\mathbf{g},\mathbf{g}))^{-1}\cdot(A(\theta))^{-1}\cdot Cov\left(\frac{\nabla_\theta p(\theta)}{p(\theta)}-A(\theta)\cdot\mathbf{g},h-\boldsymbol{\beta}_{np}\cdot\mathbf{g}\right).\end{aligned}$$

The difference depends on the covariance of the residuals of $\frac{\nabla_\theta p(\theta)}{p(\theta)}$ *and* h *with respect to projections on the space generated by the cash-flows of the input instruments.*

In particular, the difference between the two sets of coefficients will be small if either one or the other residual is small or if the residuals are nearly uncorrelated.

This result provides a qualitative understanding of the difference between parametric and non-parametric "Greeks". Despite the obvious difficulties in obtaining quantitative estimates from the above proposition, we note that if the reference instruments form a "reasonable" spanning set in the L^2 sense, the residuals will be small and the approximation will be good. Finally, the fact that the error is controlled by the residual *of* $\frac{\nabla_\theta p(\theta)}{p(\theta)}$ indicates that the non-parametric sensitivities can be reasonably good even if the target payoff h is poorly correlated with the input instruments. This last remark is consistent with the fact that the MC algorithm gives excellent numerical approximations to the Greeks even for out of the money options with deltas as small as 15%.

7 References

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