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Optimal Portfolio Liquidation and Macro Hedging

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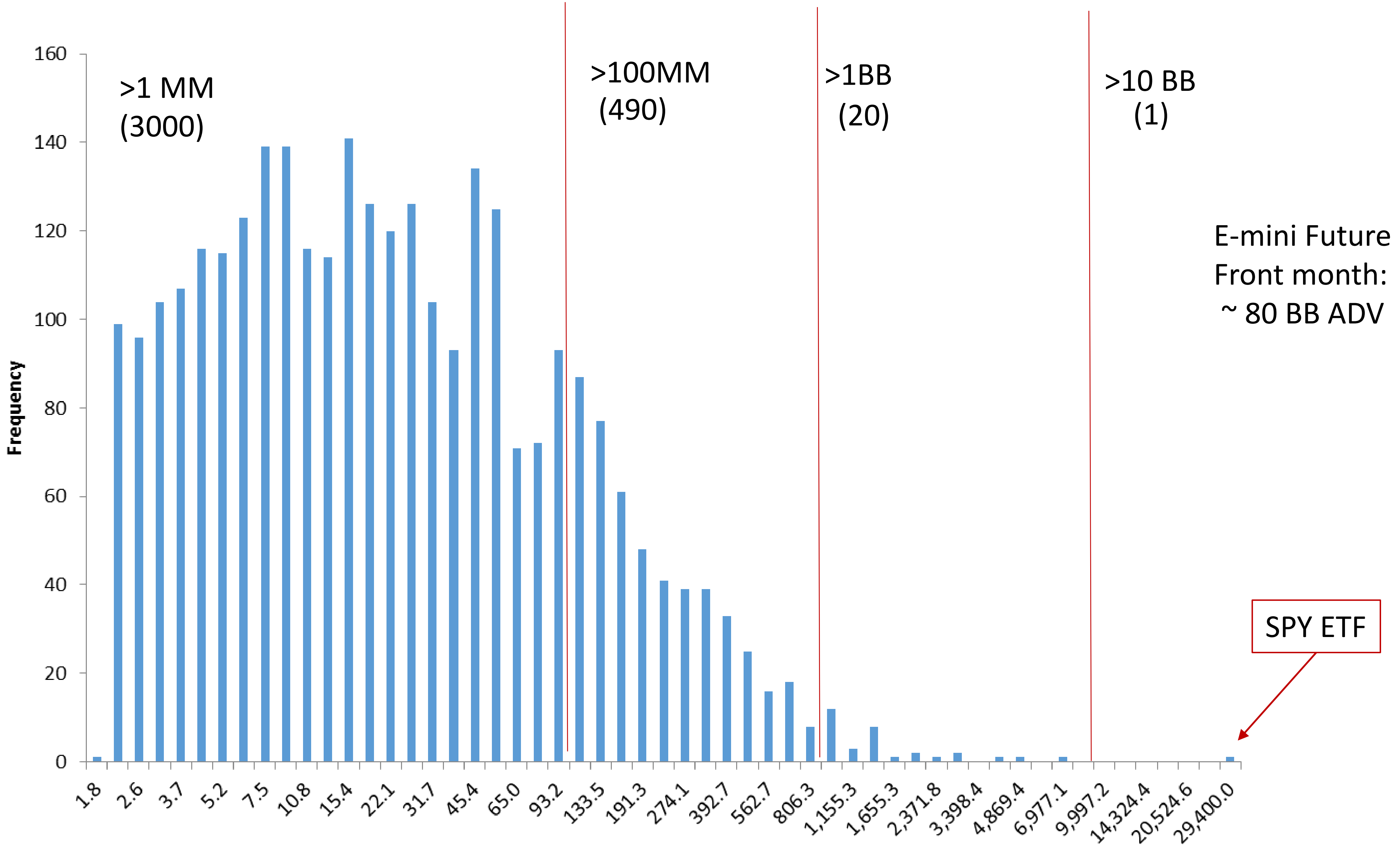
Liquidity Risk Measures

- Liquidity = the ease with which we can turn a portfolio of securities into cash.
- Equities: there is a wide dispersion in terms of bid-ask spreads and trading volumes. Listed equities are among the most liquid assets, but there are sporadically events in the market that require monitoring liquidity (e.g. ETF creation/redemption, mutual funds liquidation, flash crashes). Index futures are ``super liquid’’.
- Credit: good liquidity in indexes (CDX, ITraxx). Less so in corporate obligors.
- Government bonds: liquidity mismatch between on-the-run and off-the-run issues. Can give rise to interesting risk problems in the repo market (haircut calculations).
- Corporate bonds: trade ``by appointment only’’.
- Liquidity Measures can be a useful tool to set competitive bid-ask spreads for block trades, for IM charges for central clearing, and for risk-management of bank portfolios.

U.S. Listed Stocks: Histogram of ADV, log scale



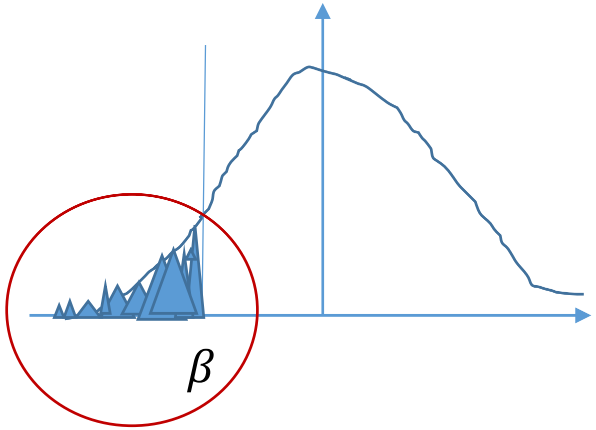
Portfolio Description & Assumptions

- N assets, with pricing functions $P_{it} = P_i(R_t)$. R_t is a vector of risk factors representing uncertainty.
- Initial quantity of asset $i = X_i$. Balance at time $t > 0 = X_{it}$.
- Each asset has an observable average daily volume (ADV), and a **liquidity threshold** $k_i = 0.1 \times ADV_i, i = 1 \dots, N$.
- Assume that a trader will transact no more than k_i units per day of asset i.
- Assume that trades can take place at price P_{it} if the liquidity restrictions are met.
- Based on these simple assumptions, we propose a **liquidity measure or liquidity charge** for portfolios.

Expected Shortfall, or CVaR

$$ES_\alpha\{Y\} := E\{Y |\, Y < \beta\}$$

where $$P\{Y < \beta\} = 1 - \alpha$$



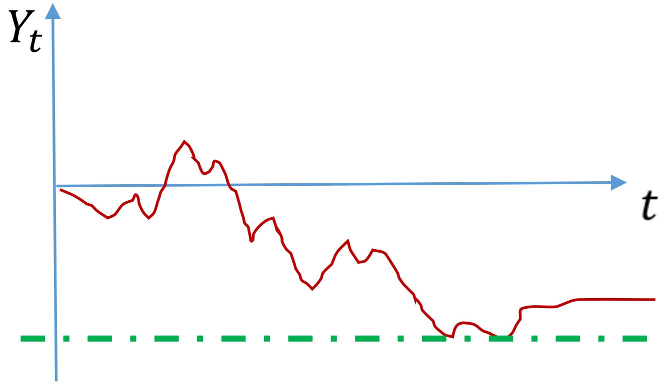
- Expected Shortfall is the average PnL conditional that the PnL is **below the Value-at-Risk** with confidence α (α=99%, 99.5%....).

- In general, we assume assets have zero drift during liquidation and that ES will be negative.

Expected Shortfall for Worst Transient Loss

- PNL at time t from a liquidation strategy:

$$Y_t = \int_0^t \sum_{i=1}^{N} X_{is} dP_{is} \qquad t > 0.$$

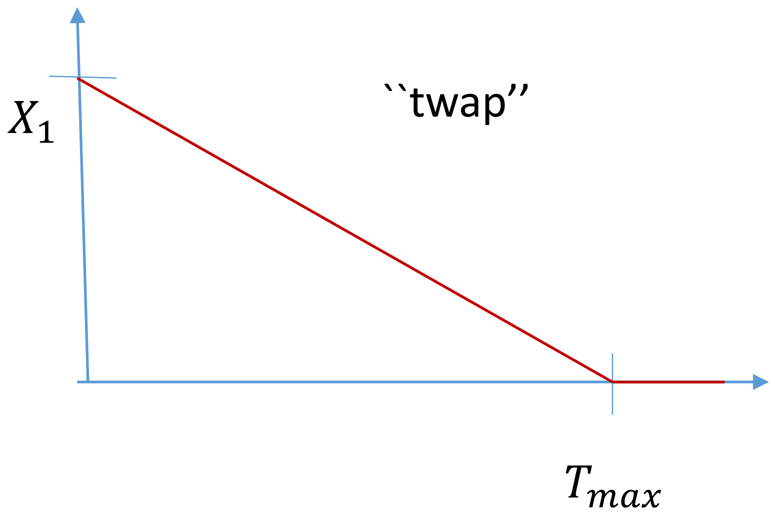


- Proposed risk measure:

$$LC(X) = -\max_{X_t \in \Omega} ES_\alpha \left\{ \min_{t>0} \int_0^t \sum_{i=1}^{N} X_{is} dP_{is} \right\}$$

$$\Omega = \left\{ X : \left| \dot{X}_{it} \right| \leq k_i ; t > 0 \right\}$$

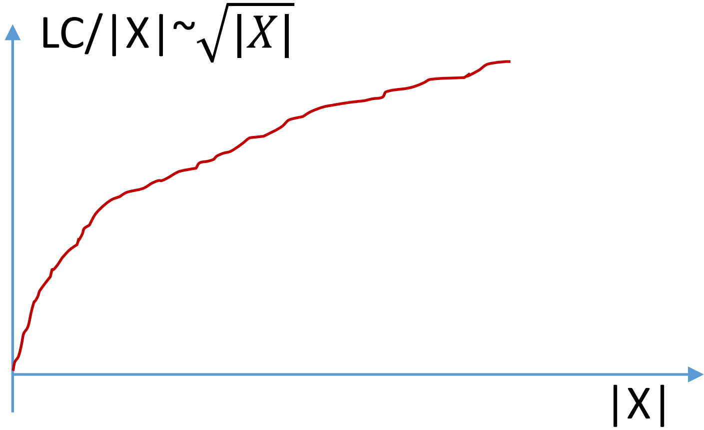
Single Asset: Optimal Execution & Liquidity Charge



Define: $$T_{max} = \frac{|X_1|}{k_1}$$

Optimal Strategy: $$X_{1t} = X_1 - sgn(X_1)\, k_1 t \quad t \leq T_{max}$$

Liquidity Charge*: $$LC(X_1) = \frac{\sigma_1}{k_1\sqrt{3}} |X_1|^{1.5}$$



* Under some additional assumptions on the asset (bounded volatility)

Portfolio Liquidity Measure

- (1) Assume, to simplify, that the assets prices are linear in the risk factors

$$\frac{dP_i}{P_i} = \sum_{j=1}^{m} \delta_{ij} dR_j = \sigma_i dZ_{it}$$

- This means that we characterize the assets by their linearized sensitivities to risk factors expressed in dollars (delta, vega, DV01,…).

- (2) Assume also that dR_j are multivariate Gaussian or Student-T.

- (3) Measure positions in dollars as opposed to contracts.

$$dY = \sum_{i=1}^{N} X_{it} \frac{dP_i}{P_i} = \sum_{i=1}^{N} X_{it} \sigma_i dZ_{it}$$

Simplified Expression for the Liquidity Charge

$$P\left\{\min_{t<T} Y_t < x\right\} = 2\,P\{Y_T < x\}$$

($\forall T$, Schwartz reflection principle)

$$ES_\alpha\left\{\min_{t<T} Y_t\right\} = ES_{\frac{1+\alpha}{2}}\{Y_T\}$$

$$= \; - \; \zeta_{\frac{1+\alpha}{2}} \; \sqrt{E\{Y_T^2\}}$$

$$LC(X) = \; \zeta_{\frac{1+\alpha}{2}} \; \min_{X_t \in \Omega} \sqrt{\int_0^\infty X_t' A X_t dt}$$

$$\Omega = \left\{X : \left|\dot{X}_{it}\right| \le k_i ; t > 0\right\}$$

A = covariance matrix

$$\zeta_{\frac{1+\alpha}{2}} = ES_{\frac{1+\alpha}{2}}\{N(0,1)\}$$

Under linearization, the problem reduces to minimizing the variance of the terminal PnL.

Constrained Linear-Quadratic Regulator

Minimize:

$$U(X) = \int_0^{\infty} X_t' A X_t dt = \int_0^{\infty} \sum_{i=1}^{N} X_{it} A_{ij} X_{jt} dt$$

subject to **initial constraints** and **velocity constraints**:

$$X_{i0} = X_i \quad \text{(initial portfolio holdings)}$$

$$\left| \dot{X}_{it} \right| \leq k_i, \quad i = 1, \ldots, N, \quad t > 0. \qquad \text{(velocity bounds)}$$

This is a non-linear problem in control theory. The non-linearity comes from the velocity constraint.

The problem can be solved by quadratic programming (QP), since it consists of minimizing a quadratic function on a convex polyhedral set.

Optimal liquidation strategies should consider hedges between different portfolio components.

Hamilton-Jacobi-Bellman Equation

- The optimal value function satisfies the HJB equation:

$$\sum_{i=1}^{N} k_i \left| \frac{\partial U}{\partial x_i} \right| = \sum_{ij=1}^{N} x_i A_{ij} x_j \,,$$

- Formally, the optimal liquidation strategy is given by

$$\dot{X}_{it} = -k_i \, Sgn \left(\frac{\partial U}{\partial x_i} (X_t) \right)$$

- Unfortunately, the HJB equation does not admit a closed-form solution and cannot be solved numerically in high dimensions. **It shows nevertheless that the trajectories are piecewise linear.**

Euler-Lagrange Equations

Euler-Lagrange:

$$\ddot{x}_{it} = k_i^2 \delta \left(\int_t^{\infty} \sum_{j=1}^{N} A_{ij} x_{js} ds \right) \sum_{j=1}^{N} A_{ij} x_{jt}$$

Pontryagin MP:

$$\dot{x}_{it} = -k_i \, Sign \left(\int_t^{\infty} \sum_{j=1}^{N} A_{ij} x_{js} ds \right)$$

(Feedback control)

We have piecewise linear trajectories that can change direction only if the sign of the quantity $\int_t^{\infty} \sum_{j=1}^{N} A_{ij} x_{js} ds$ is zero or if $\sum_{j=1}^{N} A_{ij} x_{jt}$=0.

2-D problem: Hedging Lines

Hedging lines: $L_1 \; : \; A_{11}x_1 + A_{12}x_2 = 0$

$L_2 \; : \; A_{21}x_1 + A_{22}x_2 = 0$

Stable hedging line: L_1 is stable if $k_1 \geq k_2 \frac{A_{12}}{A_{11}}$

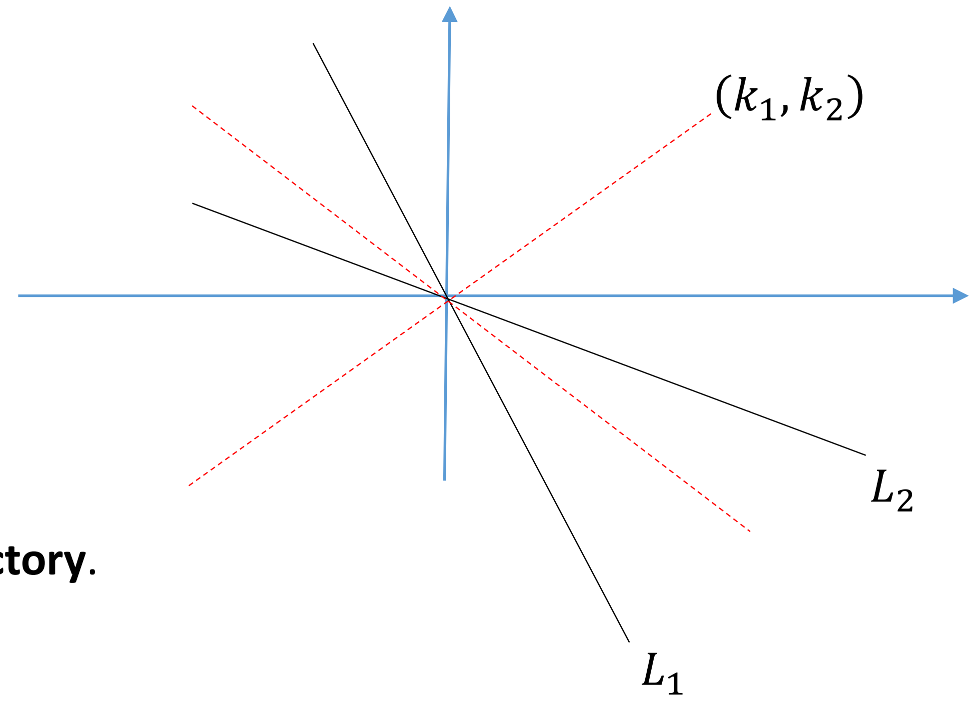
L_2 is stable if $k_2 \geq k_1 \frac{A_{12}}{A_{22}}$

Stability means that **the line is a portion of an optimal trajectory**.

Two lines stable if:

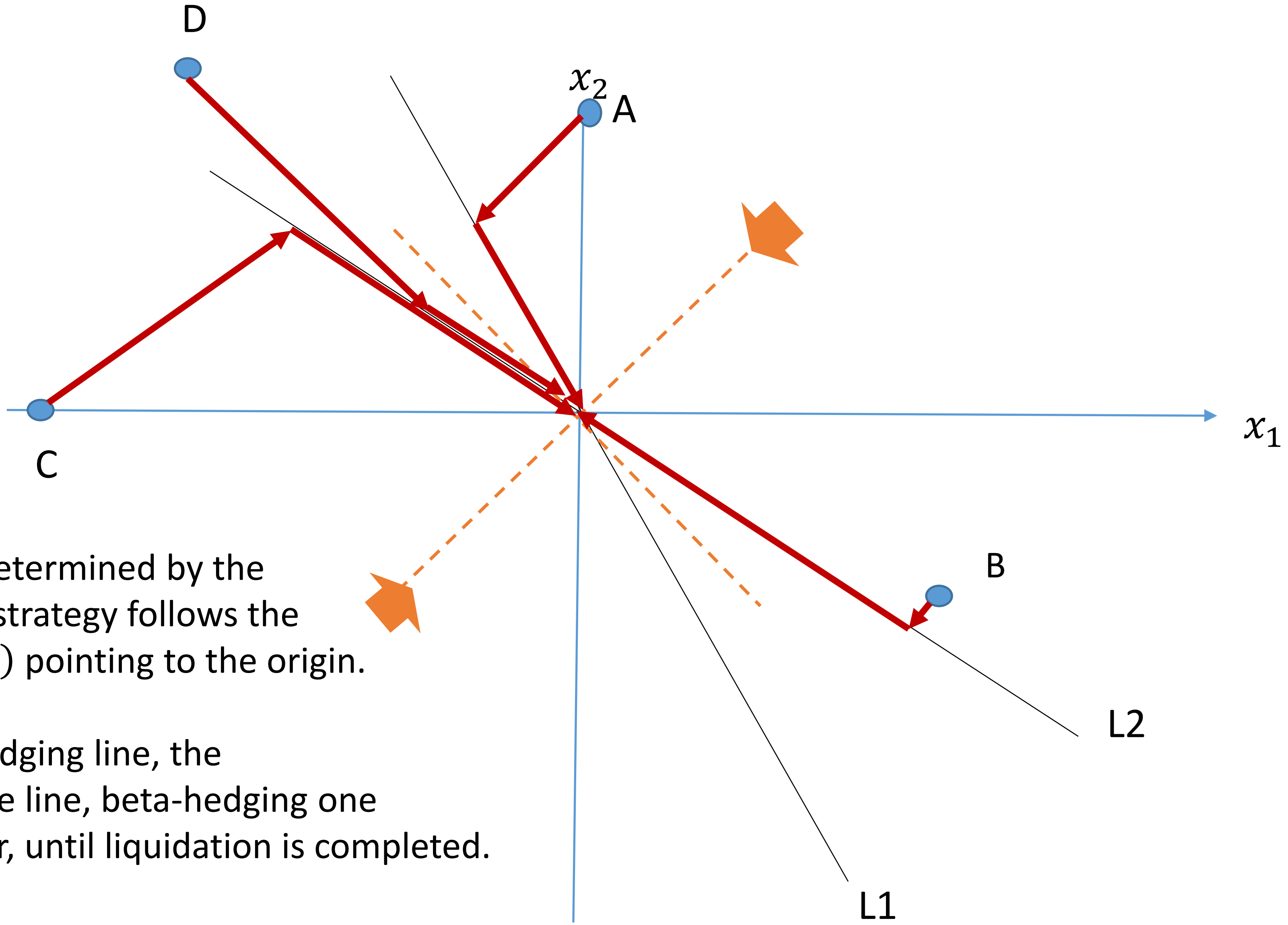
$$\rho \leq \frac{\sigma_1 k_1}{\sigma_2 k_2} \leq \frac{1}{\rho}$$

- Liquidities are comparable
- Correlation is low



Both lines are stable

Two stable hedging lines: optimal liquidation



- In each sector (determined by the hedging lines), the strategy follows the direction $(\pm k_1, \pm k_2)$ pointing to the origin.

- After hitting a hedging line, the strategy stays on the line, beta-hedging one asset with the other, until liquidation is completed.

2-D problem with only one stable line

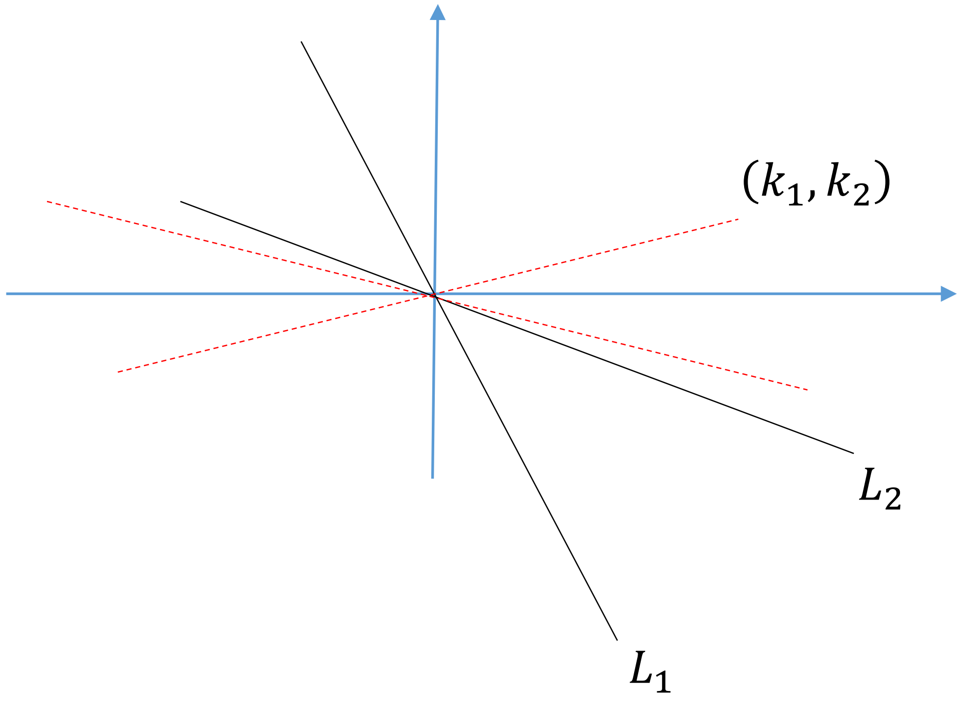
Hedging lines: $L_1 \quad : \quad A_{11}x_1 + A_{12}x_2 = 0$

$L_2 \quad : \quad A_{21}x_1 + A_{22}x_2 = 0$

Stable hedging line: L_1 is stable if $\quad k_1 \geq k_2 \frac{A_{12}}{A_{11}}$

L_2 is stable if $\quad k_2 \geq k_1 \frac{A_{12}}{A_{22}}$

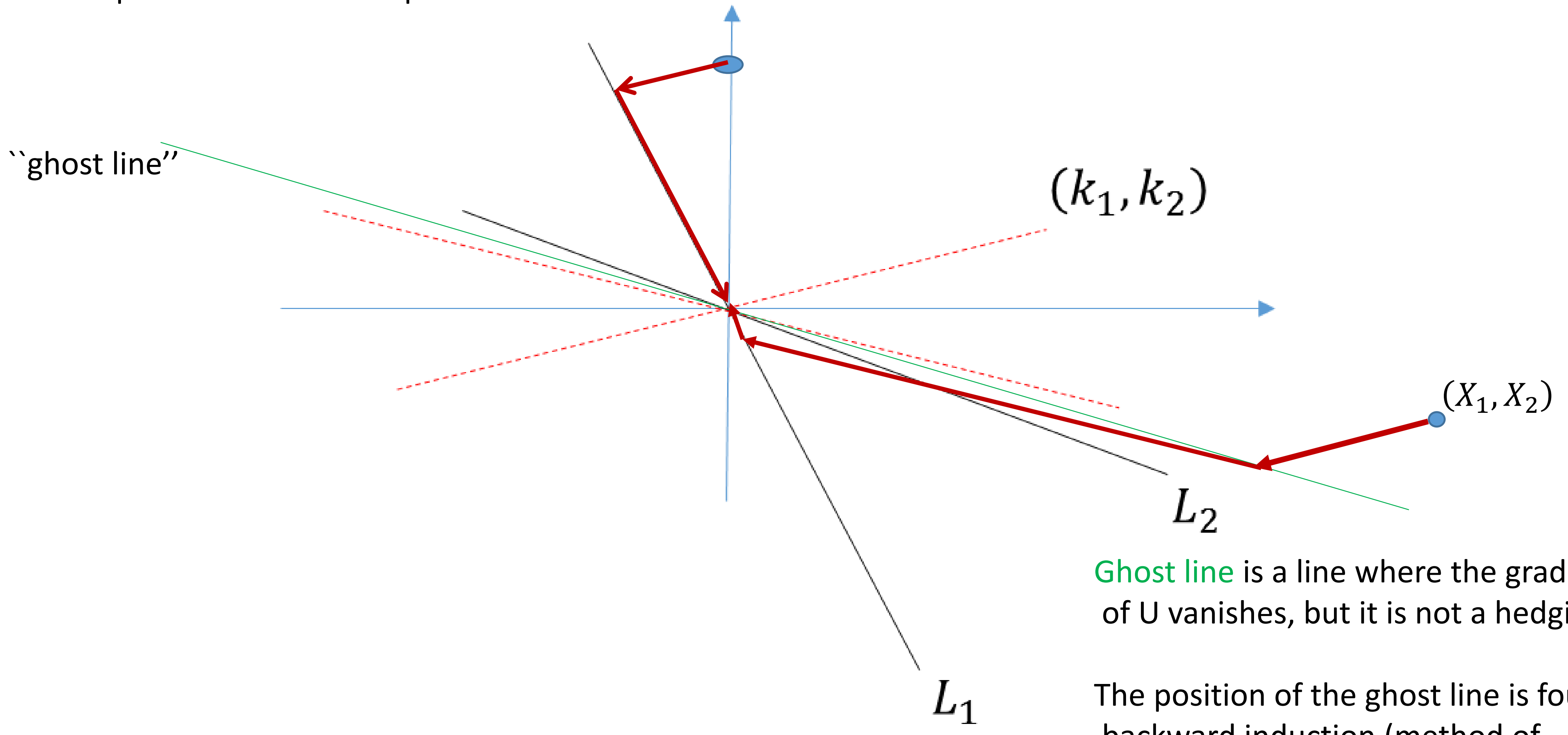
Only one hedging line corresponds to the case when the liquidities of the assets are very different



Only L_1 is stable.

Only one stable line: optimal trajectories

All trajectories flow to the stable line,
which corresponds to the most liquid asset.



Ghost line is a line where the gradient of U vanishes, but it is not a hedging line.

The position of the ghost line is found by backward induction (method of characteristics)

In higher dimensions: separation of scales

- Assume one very liquid asset and other less liquid ones: $k_1 = \frac{1}{\varepsilon}, \quad \varepsilon \ll 1.$

$$\dot{x}_{1t} = -\frac{1}{\varepsilon}\, Sign\left(\int_t^\infty \sum_{j=1}^N A_{1j} x_{js} ds\right)$$

$$\dot{x}_{it} = -k_i\, Sign\left(\int_t^\infty \sum_{j=1}^N A_{ij} x_{js} ds\right)$$

i=2,3,…,N

- As $\varepsilon \longrightarrow 0$, the optimal trajectory will travel in time O(ε) to the hyperplane

$$\sum_{j=1}^N A_{1j} x_j = 0$$

and remain there until the end of the liquidation period.

Separation of Scales and Macro-hedging

- Solving for x_{1t}, we find that in time $O(\varepsilon)$ the optimal trajectory reaches the ``hedging hyperplane''

$$x_{1t} = -\frac{1}{A_{11}} \sum_{j=2}^{N} A_{1j} x_{jt} = -\sum_{j=2}^{N} \beta_{1j} x_{jt}$$

- Substituting this value in the other equations, we find that

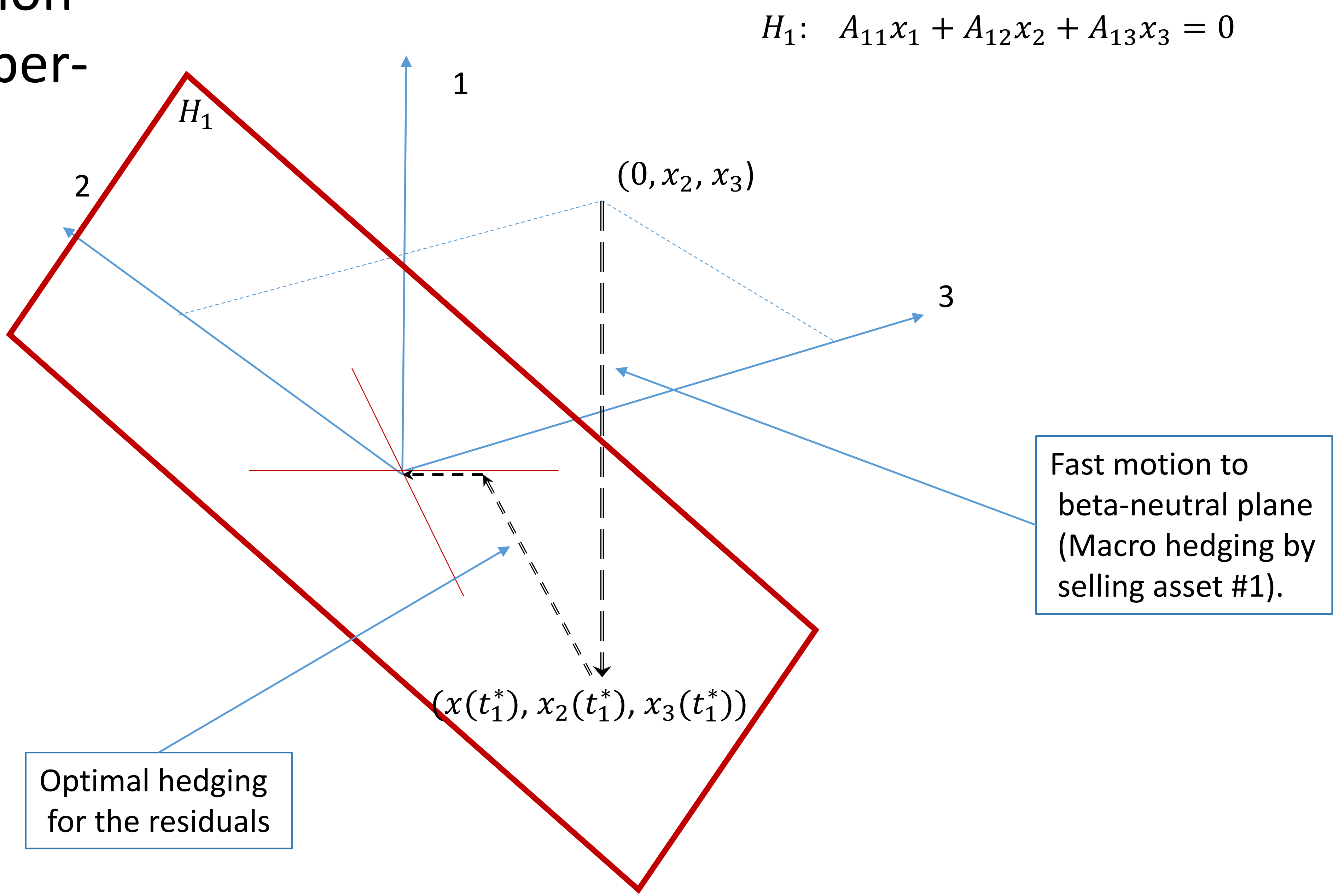
$$\dot{x}_{it} = -k_i \, Sign\left(\int_t^{\infty} \sum_{j=2}^{N} \tilde{A}_{ij} x_{js} ds \right) \qquad i = 2,3, \dots, N.$$

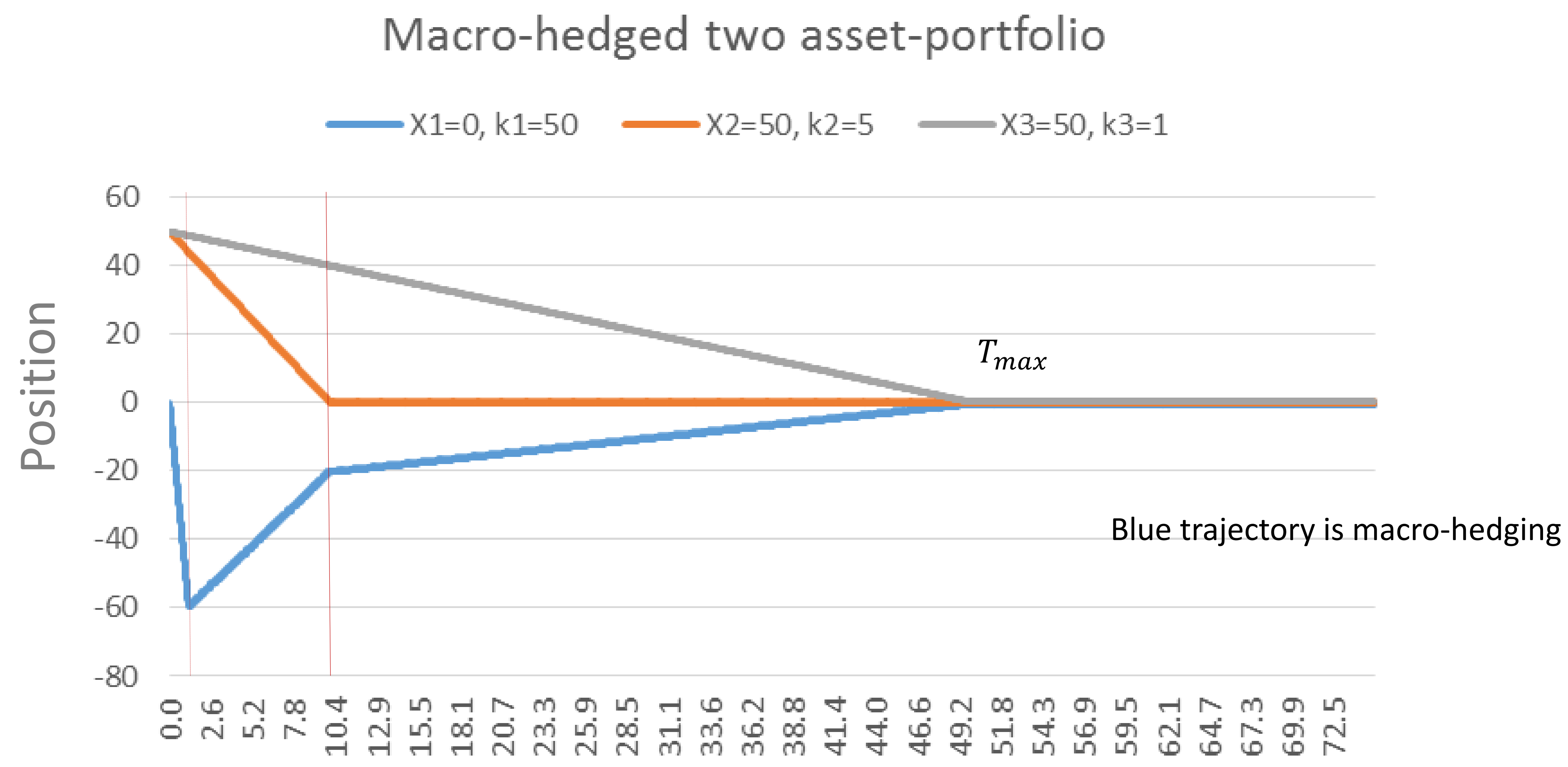
$$\tilde{A}_{ij} = A_{ij} - \frac{A_{i1} A_{j1}}{A_{11}}$$

Matrix of residuals after beta hedging with the ultra-liquid asset

In the presence of an ultra-liquid asset (e.g. index futures) it is optimal to first beta-hedge the portfolio with respect to this asset, and then proceed to liquidate optimally the ``residuals''.

3-D Liquidation with one hyper-liquid asset




Macro-hedged two asset-portfolio
X1=0, k1=50
X2=50, k2=5
X3=50, k3=1
Position
60
40
20
0
-20
-40
-60
-80
0.0
2.6
5.2
7.8
10.4
12.9
15.5
18.1
20.7
23.3
25.9
28.5
31.1
33.6
36.2
38.8
41.4
44.0
46.6
49.2
51.8
54.3
56.9
59.5
62.1
64.7
67.3
69.9
72.5
T_{max}
Blue trajectory is macro-hedging

Questions

1. How efficient is macro-hedging (hedging with index derivatives) in the context of optimal liquidation?

2. Does separation of scales work in practice?

3. Find an approximation to the LC which does not require (if possible) solving the full problem.

More tractable problem: the Almgren-Chriss/ Garleanu-Pedersen LQR models

Replace ``hard'' liquidity constraint by quadratic penalty.

$$U_{ac}(X) = \min_{X \in \Xi} \int_0^\infty \left[X_t' A X_t + \sum_{i=1}^{N} \left| \frac{\dot{X}_{it}}{k_i} \right|^2 \right] dt$$

Quadratic velocity penalty

$$\Xi = \{X : X_{i0} = X_0\}$$

- Almgren-Chriss model (2000) uses quadratic market impact functions for optimal execution; see also Garleanu and Pedersen (2010) for portfolio management with transaction costs.
- LQR **do not have the desirable 3/2-power law,** as $\sqrt{U_{ac}(X)}$ scales linearly with portfolio size.
- Nevertheless, multi-D LQR model is fully tractable and thus useful to test separation of scales.

Hamilton-Jacobi-Bellman equation for LQR

- The equation is: $$\sum_{i=1}^{N} k_i^2 \left(\frac{\partial U}{\partial x_i}\right)^2 = x' A x$$

- The solution is:

$$U(x) = x' M x \qquad M = L^{-1}(LAL)^{1/2}L^{-1}$$

where

$$L = diag(k_1,\ldots,k_N)$$

- The optimal strategy is: $$\dot{X}_t = -K X_t\,, \qquad K = L(LAL)^{1/2}L^{-1}$$

In 1 D,

$$X_t = X_0 e^{-k\sigma t}$$

Empirical Study

- We considered the ~ 500 stocks composing the S&P 500 index and the E-mini S&P Index futures.
- We constructed 100 portfolios of 20 randomly chosen stocks among the 500 stocks. (Long only).
- For each portfolio we computed:

1. The cost of liquidating the positions using the 1-D LQR model separately on each position.
2. The cost of liquidating optimally the portfolio using the multi-D LQR strategy
3. The cost of liquidating optimally the portfolio, including E-mini S&P, using the multi-D LQR strategy
4. The sum of (A) the costs of the macro-hedge and (B) the cost of liquidating optimally the residuals
5. The sum of (A) the cost of the macro hedge and (B) to liquidate the residuals using the 1D LQR for each asset.

Formulas for various costs (assume $\zeta = 1$)

LQR, with and without MH

$$\left(LC_{lqr}\right)^2 = X'MX \qquad M = L^{-1}(LAL)^{1/2}L^{-1}$$

Separation of Scales: MH followed by LQR on residuals

$$(LC_{ss})^2 = \left(X_1 - \sum_{j=2}^{N} \beta_{1j} X_j\right)^2 \frac{\sigma_1}{k_1} + \widetilde{X}'\widetilde{M}\widetilde{X} \qquad \tilde{X} = (0, X_2, \ldots, X_N)'$$

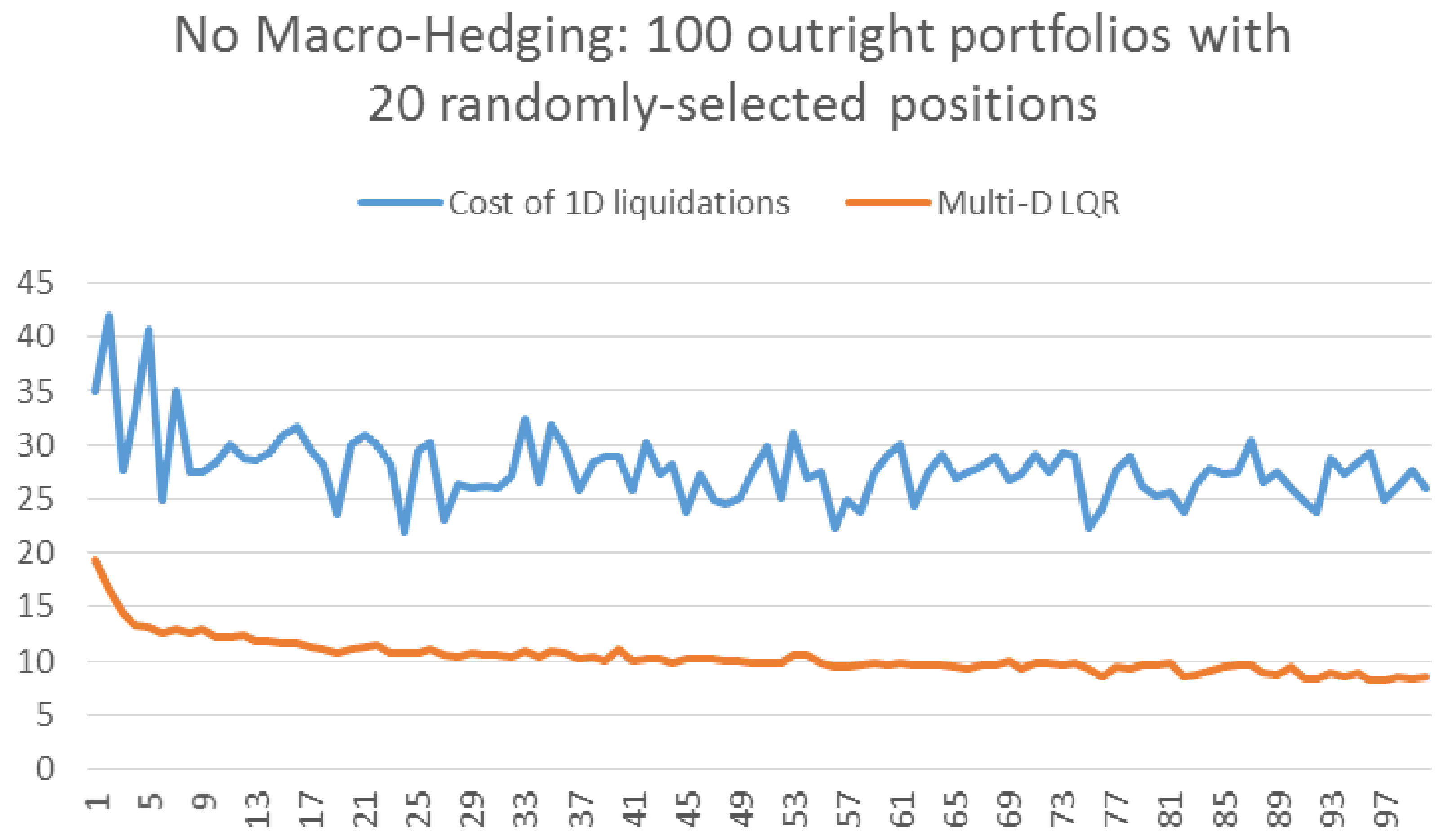
Naïve strategy: 1-D LQR liquidations

$$(LC_{1D})^2 = \sum_{mn=1}^{N} \frac{A_{mn} X_m X_n}{\sigma_m k_m + \sigma_n k_n}$$

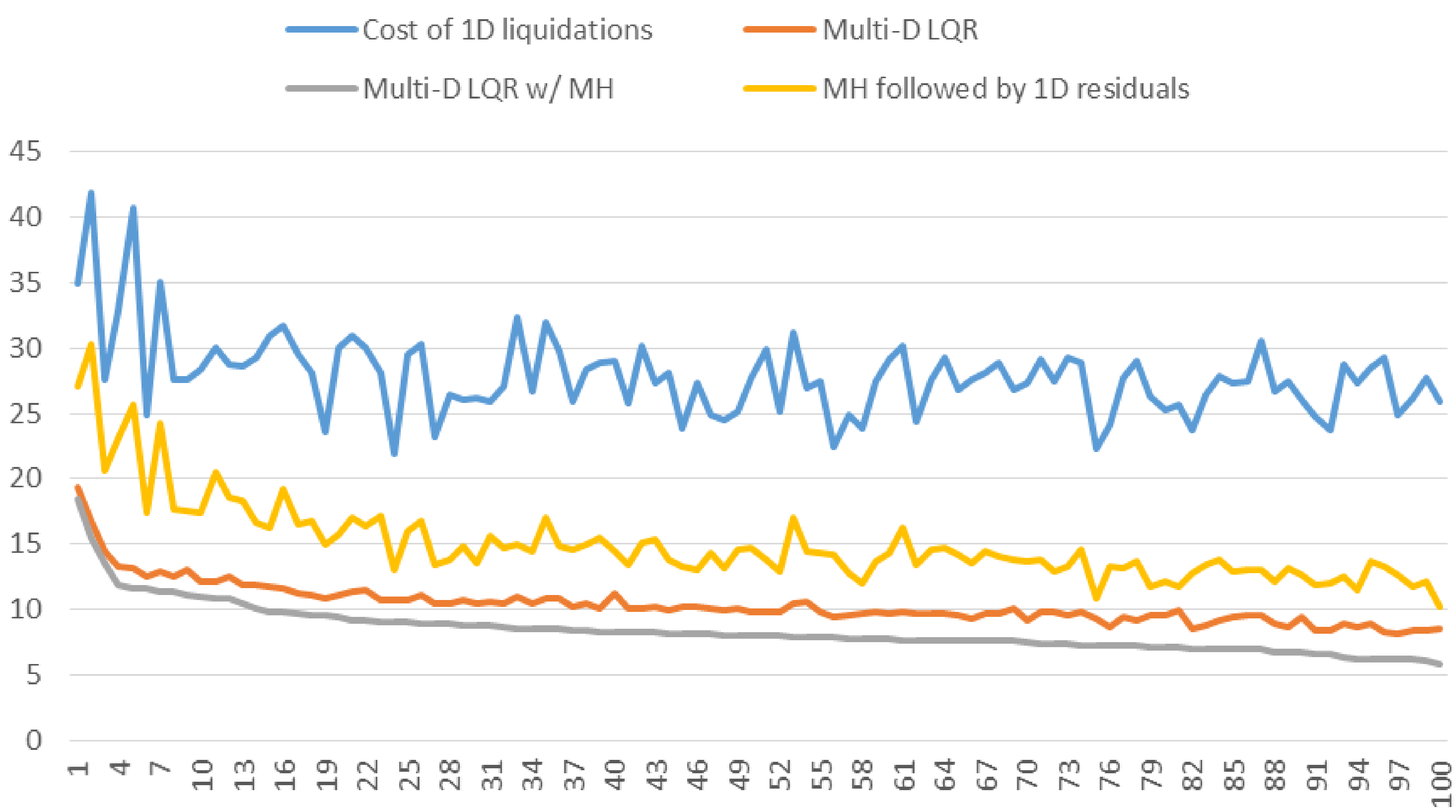
Macro-hedging followed by naïve strategy on residuals

$$(LC_{MH+1D})^2 = \left(X_1 - \sum_{j=2}^{N} \beta_{1j} X_j\right)^2 \frac{\sigma_1}{k_1} + \sum_{mn=2}^{N} \frac{\tilde{A}_{mn} X_m X_n}{\sigma_m k_m + \sigma_n k_n}$$

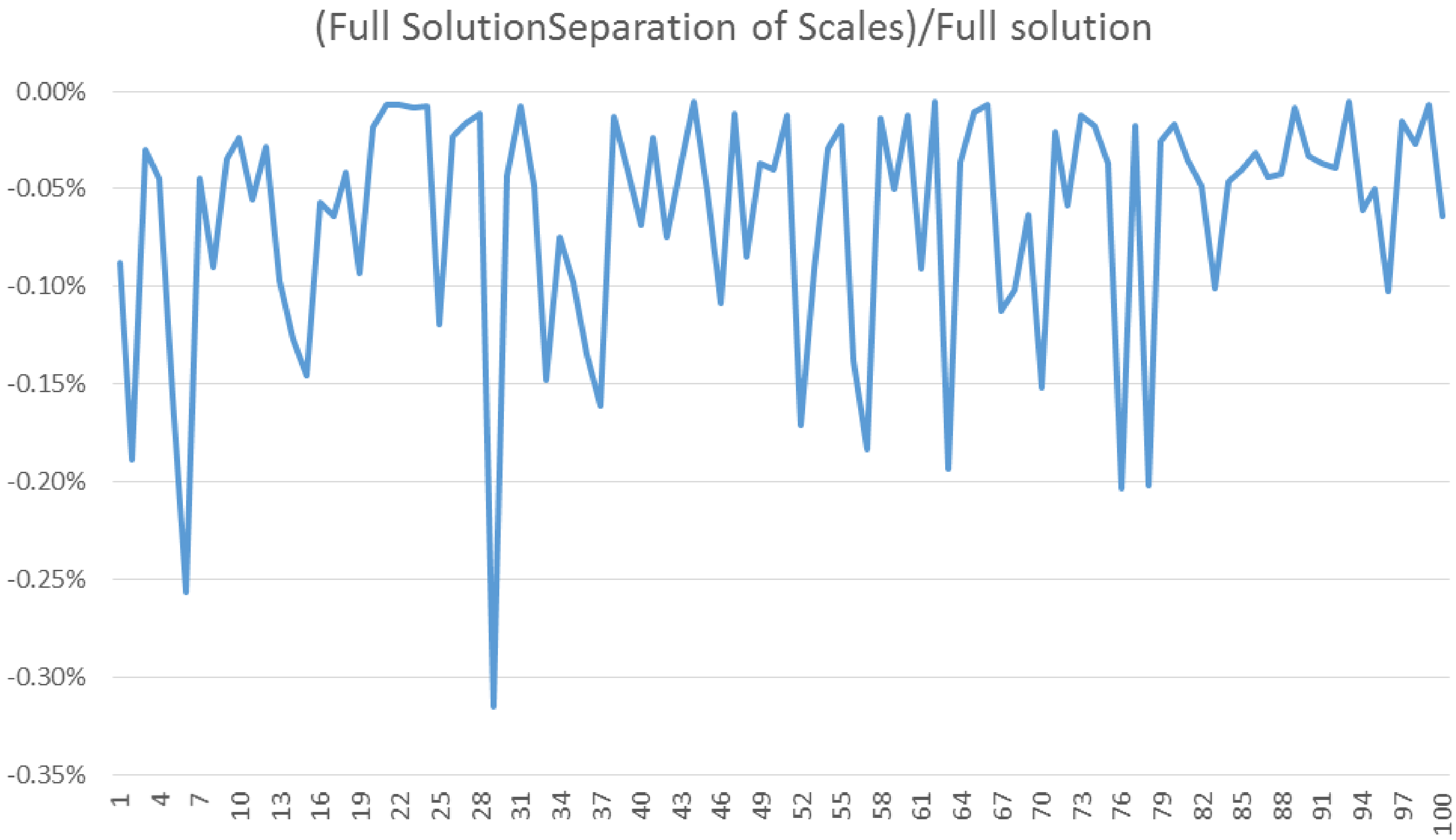
Testing randomly-generated portfolios using S&P constituents



Effect of Macro-hedging on LQR and Naïve strategies



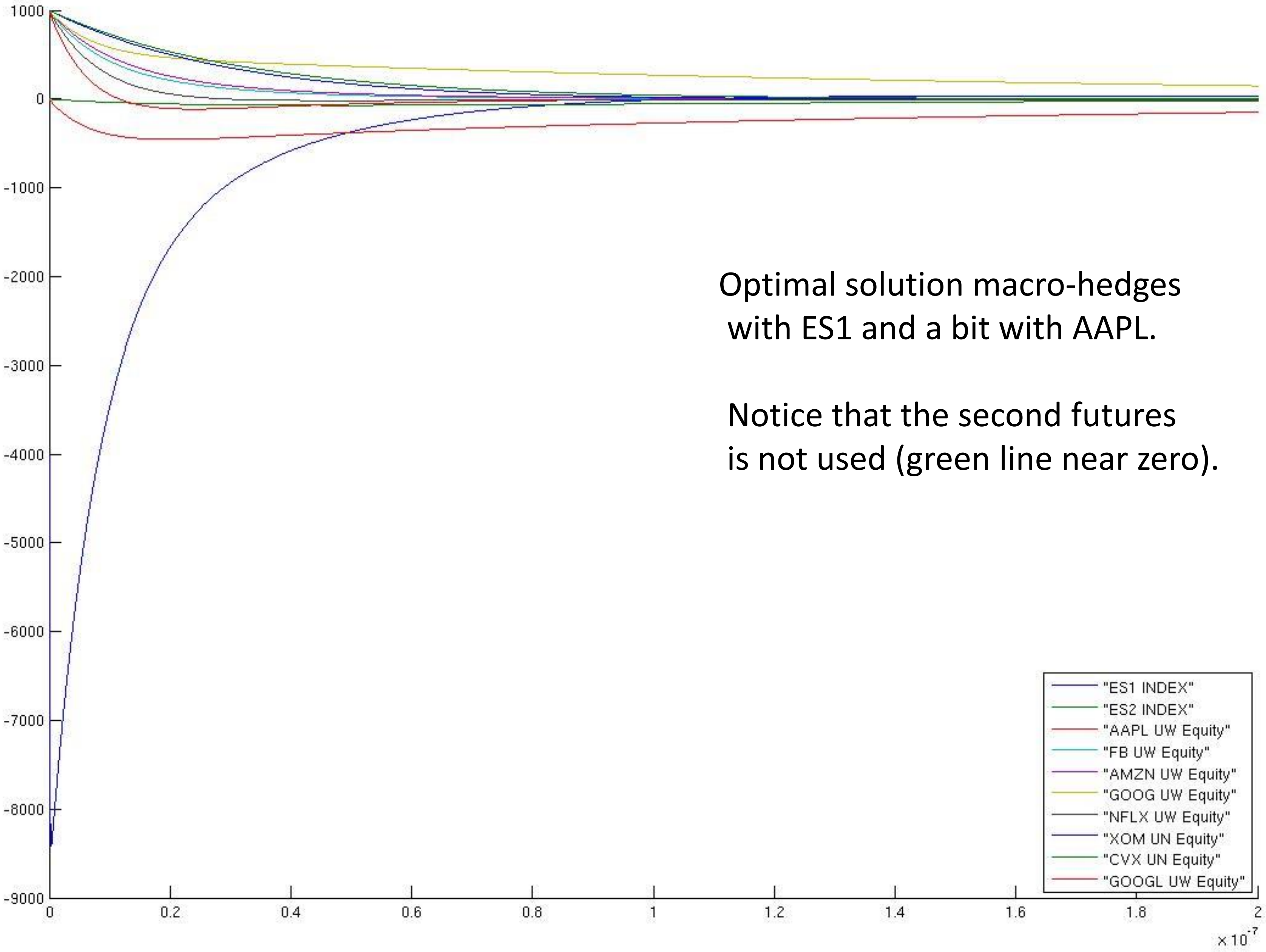
Costs: Comparing exact LQR vs. Separation of Scales



Separation of scales approximation is very close to LQR. (Here we solve the residuals problem exactly with LQR).

Small portfolio:
Macro hedging
with e-mini S&P
futures

For stocks:
only the first
contract is useful.

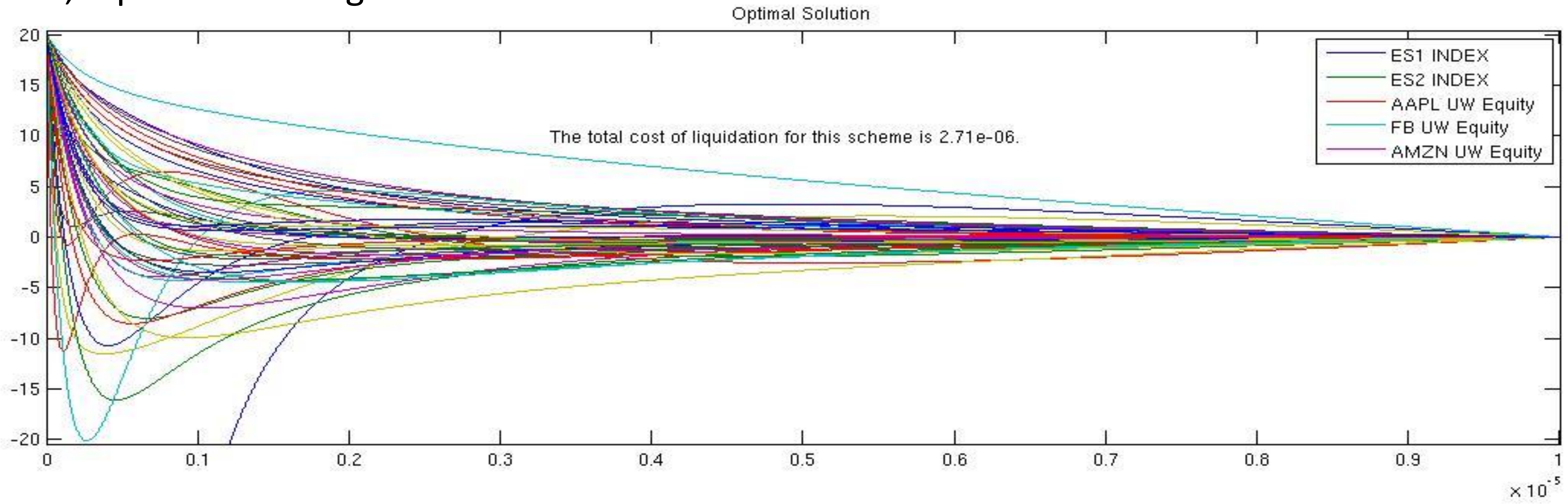


Optimal solution macro-hedges with ES1 and a bit with AAPL.

Notice that the second futures is not used (green line near zero).

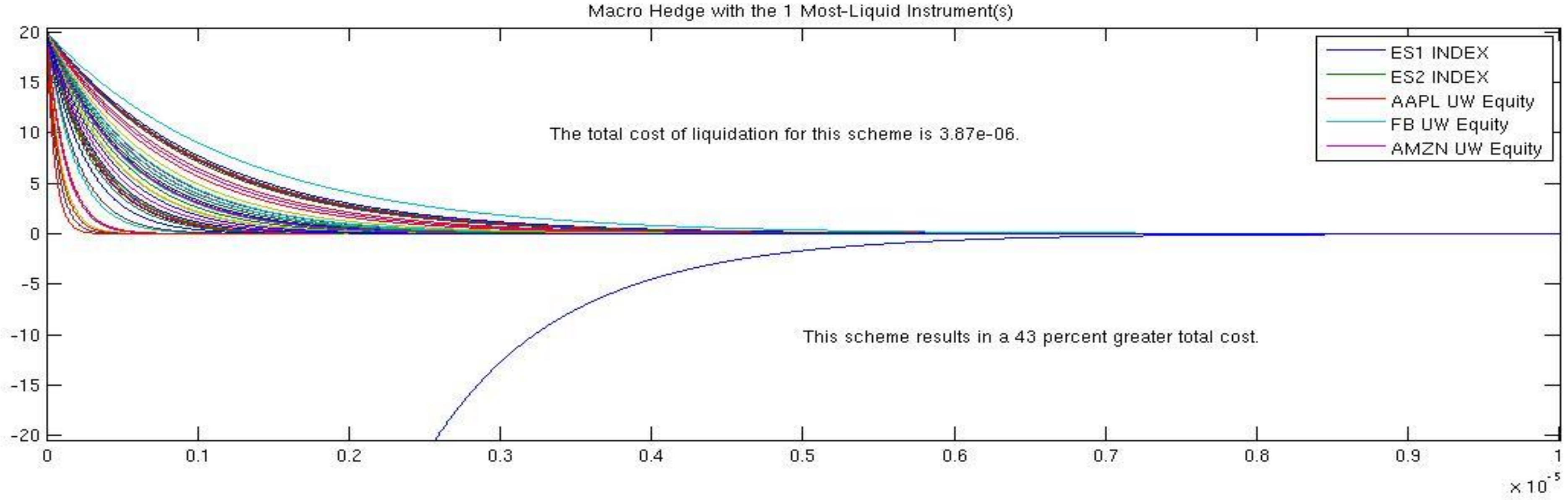
500 stock portfolio, equal dollar weighted

Full LQR



MH +

Naïve Liq of residuals



Concluding remarks

- This approach to Liquidity charges has been applied to Credit markets, Equity Derivatives clearing & U.S. Treasury bonds.

- Markets change and calibrations may be different, but the mathematical framework and approximations (MH+Naïve…, etc.) are very similar.

- LQR is a good tool to explore the properties of optimal paths, but it is unrealistic in some ways.

- The`` liquidity constraints’’ approach gives simpler strategies and has the correct scaling. But it is also harder to solve.

- The **constrained problem with full-valuation** is similar to BM&F Bovespa’s CORE, which is solved by linear programming, but can be computationally expensive for large books.